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Anomaly and Exotic Statistics in One Dimension

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ABSTRACT

We study the influence of the anomaly on the physical quantum picture of the chiral Schwinger model (CSM) defined on S^1. We show that such phenomena as the total screening of charges and the dynamical mass generation characteristic for the Schwinger model do not take place here. Instead of them, the anomaly results in the background linearly rising electric field or, equivalently, in the exotic statistics of the physical matter field. We construct the algebra of the Poincare generators and show that it differs from the Poincare one. For the CSM on R^1, the anomaly influences only the mass generation mechanism.

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1 INTRODUCTION

The two-dimensional QED with massless fermions, i.e. the Schwinger model (SM), demonstrates such phenomena as the dynamical mass generation and the total screening of the charge [1] . Although the Lagrangian of the SM contains only massless fields, a massive boson field emerges out of the interplay of the dynamics that govern the original fields. This mass generation is due to the complete compensation of any charge inserted into the vacuum.

In the chiral Schwinger model [2, 3] only the right (or left) chiral component of the fermionic field is coupled to the $U(1)$ gauge field. The left-right asymmetric matter content leads to an anomaly. At the quantum level, the local gauge symmetry is not realized by a unitary action of the gauge symmetry group on Hilbert space. The Hilbert space furnishes a projective representation of the symmetry group [4, 5, 6].

In this paper, we aim to study the influence of the anomaly on the physical quantum picture of the CSM. Do the dynamical mass generation and the total screening of charges take place also in the CSM? Are there any new physical effects caused just by the left-right asymmetry? These are the questions which we want to answer.

To get the physical quantum picture of the CSM we need first to construct a self-consistent quantum theory of the model and then solve all the quantum constraints. In the quantization procedure, the anomaly manifests itself through a special Schwinger term in the commutator algebra of the Gauss law generators. This term changes the nature of the Gauss law constraint: instead of being first-class constraint, it turns into second-class one. As a consequence, the physical quantum states cannot be defined as annihilated by the Gauss law generator.

There are different approaches to overcome this problem and to consistently quantize the CSM. The fact that the second class constraint appears only after quantization means that the number of degrees of freedom of the quantum theory is larger than that of the classical theory. To keep the Gauss law constraint first-class, Faddeev and Shatashvili proposed adding an auxiliary field in such a way that the dynamical content of the model does not change [7]. At the same time, after quantization it is the auxiliary field that furnishes the additional "irrelevant" quantum degrees of freedom. The auxiliary field is described by the Wess-Zumino term. When this term is added to the Lagrangian of the original model, a new, anomaly-free model is obtained. Subsequent canonical quantization of the new model is achieved by the Dirac procedure.

For the CSM, the correspondig WZ-term is not defined uniquely. It contains the so called Jackiw-Rajaraman parameter $a > 1$. This parameter reflects an ambiguity in the bosonization procedure and in the construction of the WZ-term. Although the spectrum of the new, anomaly-free model turns out to be relativistic and contains a relativistic boson, the mass of the boson also depends on the Jackiw-Rajaraman parameter [2, 3]. This mass is definetely unphysical and corresponds to the unphysical degrees of freedom. The quantum theory containing such a parameter in the spectrum is not consistent or, at least, is not that final version of the quantum theory which we would like to get.

In another approach also formulated by Faddeev [8], the auxiliary field is not added,

so the quantum Gauss law constraint remains second-class. The standard Gauss law is assumed to be regained as a statement valid in matrix elements between some states of the total Hilbert space, and it is the states that are called physical. The theory is regularized in such a way that the quantum Hamiltonian commutes with the nonmodified, i.e. second-class quantum Gauss law constraint. The spectrum is non-relativistic [9, 10].

Here, we follow the approach given in our previous work [11, 12]. The pecularity of the CSM is that its anomalous behaviour is trivial in the sense that the second class constraint which appears after quantization can be turned into first class by a simple redefinition of the canonical variables. This allows us to formulate a modified Gauss law to constrain physical states. The physical states are gauge-invariant up to a phase. In [13, 14, 15], the modification of the Gauss law constraint is obtained by making use of the adiabatic approach.

Contrary to [11, 12] where the CSM is defined on R^1 , we suppose here that space is a circle of length L, $-\frac{L}{2} \leq x < \frac{L}{2}$, so space-time manifold is a cylinder $S^1 \times R^1$. The gauge field then acquires a global physical degree of freedom represented by the non-integrable phase of the Wilson integral on S^1. We show that this brings in the physical quantum picture new features of principle.

Another way of making two-dimensional gauge field dynamics nontrivial is by fixing the spatial asymptotics of the gauge field [16, 17]. If we assume that the gauge field defined on R^1 diminishes rather rapidly at spatial infinities, then it again acquires a global physical degree of freedom. We will see that the physical quantum picture for the model defined on S^1 is equivalent to that obtained in [16, 17].

We work in the temporal gauge $A_0 = 0$ in the framework of the canonical quantization scheme and use the Dirac's quantization method for the constrained systems [18]. In Section 2, we quantize our model in two steps. First, the matter fields are quantized, while A_1 is handled as a classical background field. The gauge field A_1 is quantized afterwords, using the functional Schrodinger representation. We derive the anomalous commutators with nonvanishing Schwinger terms which indicate that our model is anomalous.

In Section 3, we show that the Schwinger term in the commutator of the Gauss law generators is removed by a redefinition of these generators and formulate the modified quantum Gauss law constraint. We prove that this constraint can be also obtained by using the adiabatic approximation and the notion of quantum holonomy.

In Section 4, we construct the physical quantum Hamiltonian consistent with the modified quantum Gauss law constraint, i.e. invariant under the modified gauge transformations both topologically trivial and non-trivial. We introduce the modified topologically non-trivial gauge transformation operator and define θ–states which are its eigenstates. We define the exotic statistics matter field and reformulate the quantum theory in terms of this field.

In Section 5, we construct two other Poincare generators, i.e. the momentum and the boost. We act in the same way as before with the Hamiltonian, namely we define the physical generators as those which are invariant under both topologically trivial and non-trivial gauge transformations. We show that the algebra of the constructed generators is not a Poincare one.

In Section 6, we study the charge screening. We introduce external charges and calculate (i) the energy of the ground state of the physical Hamiltonian with the external charges and (ii) the current density induced by these charges.

Section 7 contains our conclusions and discussion.

2 QUANTIZATION PROCEDURE

2.1 CLASSICAL THEORY

The Lagrangian density of the CSM is

$$\mathcal{L} = -\frac{1}{4}\mathrm{F}_{\mu\nu}\mathrm{F}^{\mu\nu} + \bar{\psi} i\hbar\gamma^{\mu}\partial_{\mu}\psi + e\hbar\bar{\psi}_{\mathrm{R}}\gamma^{\mu}\psi_{\mathrm{R}}A_{\mu}, \tag{1}$$

where $\mathrm{F}_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$, $(\mu,\nu) = \overline{0,1}$, γ^{μ} are (2×2)–Dirac matrices:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The field ψ is 2–component Dirac spinor, $\bar{\psi} = \psi^{\star}\gamma^0$ and $\psi_{\mathrm{R}} = \frac{1}{2}(1+\gamma^5)\psi$.

In the temporal gauge $A_0 = 0$, the Hamiltonian density without the left-handed matter part is

$$\begin{aligned} \mathcal{H}_{\mathrm{R}} &= \mathcal{H}_{\mathrm{EM}} + \mathcal{H}_{\mathrm{F}}, \\ \mathcal{H}_{\mathrm{EM}} &= \frac{1}{2}\mathrm{E}^2, \\ \mathcal{H}_{\mathrm{F}} &\equiv \hbar\psi_{\mathrm{R}}^{\star} d\psi_{\mathrm{R}} = \hbar\psi_{\mathrm{R}}^{\star}(-i\partial_1 - eA_1)\psi_{\mathrm{R}}, \end{aligned} \tag{2}$$

where E is a momentum canonically conjugate to A_1.

On the circle boundary conditions for the fields must be specified. We impose the following ones

$$\begin{aligned} A_1(-\frac{\mathrm{L}}{2}) &= A_1(\frac{\mathrm{L}}{2}) \\ \psi_{\mathrm{R}}(-\frac{\mathrm{L}}{2}) &= \psi_{\mathrm{R}}(\frac{\mathrm{L}}{2}). \end{aligned} \tag{3}$$

The Lagrangian density 1 and the Hamiltonian density 2 are invariant under local time-independent transformations

$$\begin{aligned} A_1 &\to A_1 + \partial_1\lambda, \\ \psi_{\mathrm{R}} &\to \exp\{ie\lambda\}\psi_{\mathrm{R}}, \end{aligned}$$

generated by

$$\mathrm{G} = \partial_1\mathrm{E} + ej_{\mathrm{R}},$$

as well as under global gauge transformations of the right-handed Dirac field which are generated by

$$\mathrm{Q}_{\mathrm{R}} = \int_{-\mathrm{L}/2}^{\mathrm{L}/2} dx\, j_{\mathrm{R}}(x),$$

where $j_{\mathrm{R}} = \hbar\psi_{\mathrm{R}}^{\star}\psi_{\mathrm{R}}$ is the classical right-handed fermionic current and $\lambda(x)$ is a gauge function.

Due to the gauge invariance, the Hamiltonian density is not unique. On the constrained submanifold $\mathrm{G} \approx 0$ of the full phase space, the Hamiltonian density

$$\mathcal{H}_{\mathrm{R}}^{\star} = \mathcal{H}_{\mathrm{R}} + v_{\mathrm{H}} \cdot \mathrm{G}, \tag{4}$$

where v_{H} is an arbitrary Lagrange multiplier depending generally on field variables and their momenta, reduces to the Hamiltonian density $\mathcal{H}_{\mathrm{R}}$. In this sense, there is no difference between $\mathcal{H}_{\mathrm{R}}$ and $\mathcal{H}_{\mathrm{R}}^{\star}$, and so both Hamiltonian densities are physically equivalent to each other.

The gauge transformations which respect the boundary conditions 3 must be of the form

$$\lambda(\frac{\mathrm{L}}{2}) = \lambda(-\frac{\mathrm{L}}{2}) + \frac{2\pi}{e} n, \quad \mathrm{n} \in \mathcal{Z}. \tag{5}$$

We see that the gauge transformations under consideration are divided into topological classes characterized by the integer n. If $\lambda(\frac{\mathrm{L}}{2}) = \lambda(-\frac{\mathrm{L}}{2})$, then the gauge transformation is topologically trivial and belongs to the $n = 0$ class. If $n \neq 0$ it is nontrivial and has winding number n.

Given Eq. 5, the nonintegrable phase

$$\Gamma(A) = \exp\{\frac{ie}{2\pi} \int_{-\mathrm{L}/2}^{\mathrm{L}/2} dx A_1(x,t)\}$$

is a unique gauge-invariant quantity that can be constructed from the gauge field [19, 20, 21, 22]. By a topologically trivial transformation we can make A_1 independent of x,

$$A_1(x,t) = b(t),$$

i.e. obeying the Coulomb gauge $\partial_1 A_1 = 0$, then

$$\Gamma(A) = \exp\{i\frac{e\mathrm{L}}{2\pi} b(t)\}.$$

In contrast to $\Gamma(A)$, the line integral

$$b(t) = \frac{1}{\mathrm{L}} \int_{-\mathrm{L}/2}^{\mathrm{L}/2} dx A_1(x,t)$$

is invariant only under the topologically trivial gauge transformations. The gauge transformations from the nth topological class shift b by $\frac{2\pi}{e\mathrm{L}} n$. By a non-trivial gauge transformation of the form $g_n = \exp\{i\frac{2\pi}{\mathrm{L}} nx\}$, we can then bring b into the interval $[0, \frac{2\pi}{e\mathrm{L}}]$. The configurations $b = 0$ and $b = \frac{2\pi}{e\mathrm{L}}$ are gauge equivalent, since they are connected by the gauge transformation from the first topological class. The gauge-field configuration is therefore a circle with length $\frac{2\pi}{e\mathrm{L}}$.

2.2 QUANTIZATION AND ANOMALY

The eigenfunctions and the eigenvalues of the first quantized fermionic Hamiltonian are

$$d\langle x|n;\mathrm{R}\rangle = \varepsilon_{n,\mathrm{R}}\langle x|n;\mathrm{R}\rangle,$$

where

$$\langle x|n;\mathrm{R}\rangle = \frac{1}{\sqrt{\mathrm{L}}}\exp\{ie\int_{-\mathrm{L}/2}^{x} dz A_1(z) + i\varepsilon_{n,\mathrm{R}}\cdot x\},$$

$$\varepsilon_{n,\mathrm{R}} = \frac{2\pi}{\mathrm{L}}(n - \frac{eb\mathrm{L}}{2\pi}).$$

We see that the energy spectrum depends on b. For $\frac{eb\mathrm{L}}{2\pi}=$ integer, the spectrum contains the zero energy level. As b increases from 0 to $\frac{2\pi}{e\mathrm{L}}$, the energies of $\varepsilon_{n,\mathrm{R}}$ decrease by $\frac{2\pi}{\mathrm{L}}$. Some of energy levels change sign. However, the spectrum at the configurations $b=0$ and $b=\frac{2\pi}{e\mathrm{L}}$ is the same, namely, the integers, as it must be since these gauge-field configurations are gauge-equivalent. In what follows, we will use separately the integer and fractional parts of $\frac{eb\mathrm{L}}{2\pi}$, denoting them as $[\frac{eb\mathrm{L}}{2\pi}]$ and $\{\frac{eb\mathrm{L}}{2\pi}\}$ correspondingly.

Now we introduce the second quantized right-handed Dirac field. For the moment, we will assume that d does not have zero eigenvalue. At time $t=0$, in terms of the eigenfunctions of the first quantized fermionic Hamiltonian the second quantized (ζ–function regulated) field has the expansion [23] :

$$\psi_{\mathrm{R}}^{s}(x) = \sum_{n\in\mathcal{Z}} a_n\langle x|n;\mathrm{R}\rangle|\lambda\varepsilon_{n,\mathrm{R}}|^{-s/2}. \tag{6}$$

Here λ is an arbitrary constant with dimension of length which is necessary to make $\lambda\varepsilon_{n,\mathrm{R}}$ dimensionless, while $a_n, a_n^{\dagger}$ are right-handed fermionic creation and annihilation operators which fulfil the commutation relations

$$[a_n, a_m^{\dagger}]_+ = \delta_{m,n}.$$

For $\psi_{\mathrm{R}}^{s}(x)$, the equal time anticommutator is

$$[\psi_{\mathrm{R}}^{s}(x), \psi_{\mathrm{R}}^{\dagger s}(y)]_+ = \zeta(s,x,y), \tag{7}$$

with all other anticommutators vanishing, where

$$\zeta(s,x,y) \equiv \sum_{n\in\mathcal{Z}}\langle x|n;\mathrm{R}\rangle\langle n;\mathrm{R}|y\rangle|\lambda\varepsilon_{n,\mathrm{R}}|^{-s},$$

s being large and positive. In the limit, when the regulator is removed, i.e. $s=0$, $\zeta(s=0,x,y)=\delta(x-y)$ and Eq. 7 takes the standard form.

The vacuum state of the second quantized fermionic Hamiltonian is defined such that all negative energy levels are filled:

$$\begin{aligned} a_n|\mathrm{vac};A\rangle &= 0 \quad \text{for} \quad n > [\frac{eb\mathrm{L}}{2\pi}], \\ a_n^{\dagger}|\mathrm{vac};A\rangle &= 0 \quad \text{for} \quad n \leq [\frac{eb\mathrm{L}}{2\pi}]. \end{aligned} \tag{8}$$

i.e. the levels with energy lower than (and equal to) the energy of the level $n=[\frac{eb\mathrm{L}}{2\pi}]$ are filled and the others are empty. Excited states are constructed by operating creation operators on the Fock vacuum.

In the ζ–function regularization scheme, we define the action of the functional derivative on first quantized fermionic kets and bras by

$$\begin{aligned}
\frac{\delta}{\delta A_1(x)}|n;\mathrm{R}\rangle &= \lim_{s\to 0}\sum_{m\in\mathcal{Z}}|m;\mathrm{R}\rangle\langle m;\mathrm{R}|\frac{\delta}{\delta A_1(x)}|n;\mathrm{R}\rangle|\lambda\varepsilon_{m,\mathrm{R}}|^{-s/2},\\
\langle n;\mathrm{R}|\frac{\overleftarrow{\delta}}{\delta A_1(x)} &= \lim_{s\to 0}\sum_{m\in\mathcal{Z}}\langle n;\mathrm{R}|\frac{\overleftarrow{\delta}}{\delta A_1(x)}|m;\mathrm{R}\rangle\langle m;\mathrm{R}||\lambda\varepsilon_{m,\mathrm{R}}|^{-s/2}.
\end{aligned}$$

From 6 we get the action of $\frac{\delta}{\delta A_1(x)}$ on the operators a_n, $a_n^\dagger$ in the form

$$\begin{aligned}
\frac{\delta}{\delta A_1(x)}a_n &= -\lim_{s\to 0}\sum_{m\in\mathcal{Z}}\langle n;\mathrm{R}|\frac{\delta}{\delta A_1(x)}|m;\mathrm{R}\rangle a_m|\lambda\varepsilon_{m,\mathrm{R}}|^{-s/2},\\
\frac{\delta}{\delta A_1(x)}a_n^\dagger &= \lim_{s\to 0}\sum_{m\in\mathcal{Z}}\langle m;\mathrm{R}|\frac{\delta}{\delta A_1(x)}|n;\mathrm{R}\rangle a_m^\dagger|\lambda\varepsilon_{m,\mathrm{R}}|^{-s/2}.
\end{aligned} \tag{9}$$

Next we define the quantum right-handed fermionic current and fermionic part of the second-quantized Hamiltonian as

$$\hat{j}_{\mathrm{R}}^s(x)=\frac{1}{2}\hbar[\psi_{\mathrm{R}}^{\dagger s}(x),\psi_{\mathrm{R}}^s(x)]_- \tag{10}$$

and

$$\hat{\mathrm{H}}_{\mathrm{F}}^s=\int dx\mathcal{H}_{\mathrm{F}}^s=\frac{1}{2}\hbar\int dx(\psi_{\mathrm{R}}^{\dagger s}d\psi_{\mathrm{R}}^s-\psi_{\mathrm{R}}^s d^\star\psi_{\mathrm{R}}^{\dagger s}). \tag{11}$$

Substituting 6 into 10 and 11, we get

$$\begin{aligned}
\hat{j}_{\mathrm{R}}^s(x) &= \hbar\sum_{n\in\mathcal{Z}}\frac{1}{\mathrm{L}}\exp\{i\frac{2\pi}{\mathrm{L}}nx\}\rho_s(n),\\
\rho_s(n) &\equiv \sum_{k\in\mathcal{Z}}\frac{1}{2}[a_k^\dagger,a_{k+n}]_-\cdot|\lambda\varepsilon_{k,\mathrm{R}}|^{-s/2}|\lambda\varepsilon_{k+n,\mathrm{R}}|^{-s/2}
\end{aligned}$$

and

$$\begin{gathered}
\hat{\mathrm{H}}_{\mathrm{F}}^s=\hbar\sum_{n\in\mathcal{Z}}\frac{1}{\mathrm{L}}\exp\{i\frac{2\pi}{\mathrm{L}}nx\}\mathcal{H}_{\mathrm{F}}^s(n),\\
\mathcal{H}_{\mathrm{F}}^s(n)\equiv\mathcal{H}_0^s(n)-eb\rho_s(n),\\
\mathcal{H}_0^s(n)\equiv\frac{\pi}{\mathrm{L}}\sum_{k\in\mathcal{Z}}(2k+p)\cdot\frac{1}{2}[a_k^\dagger,a_{k+p}]_-\cdot|\lambda\varepsilon_{k,\mathrm{R}}|^{-s/2}|\lambda\varepsilon_{k+p,\mathrm{R}}|^{-s/2}.
\end{gathered} \tag{12}$$

The charge corresponding to the current $\hat{j}_{\mathrm{R}}^s(x)$ is

$$\hat{\mathrm{Q}}_{\mathrm{R}}^s=\int_{-\mathrm{L}/2}^{\mathrm{L}/2}dx\hat{j}_{\mathrm{R}}^s(x)=\hbar\rho_s(0). \tag{13}$$

With Eq. 8, we have for the vacuum expectation values:

$$\begin{aligned}\langle \text{vac}, A|\hat{j}_{\text{R}}(x)|\text{vac}, A\rangle &= -\frac{1}{2}\hbar\eta_{\text{R}}, \\ \langle \text{vac}, A|\hat{\text{H}}_{\text{F}}|\text{vac}, A\rangle &= -\frac{1}{2}\hbar\xi_{\text{R}},\end{aligned}$$

where

$$\begin{aligned}\eta_{\text{R}} &\equiv \lim_{s\to 0}\frac{1}{\text{L}}\sum_{k\in\mathcal{Z}}\text{sign}(\varepsilon_{k,\text{R}})|\lambda\varepsilon_{k,\text{R}}|^{-s}, \\ \xi_{\text{R}} &\equiv \lim_{s\to 0}\frac{1}{\lambda}\sum_{k\in\mathcal{Z}}|\lambda\varepsilon_{k,\text{R}}|^{-s+1}.\end{aligned} \tag{14}$$

The operators 10, 11 and 13 can be therefore written as

$$\begin{aligned}\hat{j}_{\text{R}}(x) &= :\hat{j}_{\text{R}}(x): -\frac{1}{2}\hbar\eta_{\text{R}}, \\ \hat{\text{Q}}_{\text{R}} &= \hbar :\rho(0): -\frac{\text{L}}{2}\hbar\eta_{\text{R}}, \\ \hat{\text{H}}_{\text{F}} &= \hat{\text{H}}_0 - eb\hbar :\rho(0): -\frac{1}{2}\hbar\xi_{\text{R}},\end{aligned} \tag{15}$$

where double dots indicate normal ordering with respect to $|\text{vac}, A\rangle$ and

$$\begin{aligned}\hat{\text{H}}_0 &= \hbar\frac{2\pi}{\text{L}}\lim_{s\to 0}\{\sum_{k>[\frac{eb\text{L}}{2\pi}]}ka_k^{\dagger}a_k|\lambda\varepsilon_{k,\text{R}}|^{-s} - \sum_{k\le[\frac{eb\text{L}}{2\pi}]}ka_ka_k^{\dagger}|\lambda\varepsilon_{k,\text{R}}|^{-s}\}, \\ :\rho(0): &\equiv \frac{1}{\hbar}\hat{\text{Q}}_{\text{R,N}} = \lim_{s\to 0}\{\sum_{k>[\frac{eb\text{L}}{2\pi}]}a_k^{\dagger}a_k|\lambda\varepsilon_{k,\text{R}}|^{-s} - \sum_{k\le[\frac{eb\text{L}}{2\pi}]}a_ka_k^{\dagger}|\lambda\varepsilon_{k,\text{R}}|^{-s}\}.\end{aligned}$$

Taking the sums in 14, we get

$$\begin{aligned}\eta_{\text{R}} &= \frac{2}{\text{L}}(\{\frac{eb\text{L}}{2\pi}\} - \frac{1}{2}), \\ \xi_{\text{R}} &= -\frac{2\pi}{\text{L}}((\{\frac{eb\text{L}}{2\pi}\} - \frac{1}{2})^2 - \frac{1}{12}).\end{aligned}$$

Both operators $:\hat{j}_{\text{R}}(x):$ and $:\hat{\text{H}}_{\text{F}}:$ are well defined when acting on finitely excited states which have only a finite number of excitations relative to the Fock vacuum.

To construct the quantized electromagnetic Hamiltonian, we first introduce the Fourier expansion for the gauge field

$$A_1(x) = b + \sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}} e^{i\frac{2\pi}{\text{L}}px}\alpha_p. \tag{16}$$

Since $A_1(x)$ is a real function, α_p satisfies

$$\alpha_p = \alpha_{-p}^{\star}.$$

The Fourier expansion for the canonical momentum conjugate to $A_1(x)$ is then

$$\hat{\mathrm{E}}(x) = \frac{1}{\mathrm{L}}\hat{\pi}_b - \frac{i}{\mathrm{L}}\hbar \sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}} e^{-i\frac{2\pi}{\mathrm{L}}px}\frac{d}{d\alpha_p},$$

where $\hat{\pi}_b \equiv -i\hbar\frac{d}{db}$. The electromagnetic part of the Hamiltonian density is

$$\hat{\mathcal{H}}_{\mathrm{EM}}(x) = \hbar\sum_{p\in\mathcal{Z}}\frac{1}{\mathrm{L}}\exp\{i\frac{2\pi}{\mathrm{L}}px\}\cdot\mathcal{H}_{\mathrm{EM}}(p),$$

where

$$\mathcal{H}_{\mathrm{EM}}(p) \equiv -\frac{1}{\mathrm{L}}\hbar\frac{d}{d\alpha_{-p}}\frac{d}{db} - \frac{1}{2\mathrm{L}}\hbar\sum_{\substack{q\in\mathcal{Z}\\ q\neq(0;p)}}\frac{d}{d\alpha_{-p+q}}\frac{d}{d\alpha_{-q}} \qquad (p\neq 0), \tag{17}$$

so the corresponding quantum Hamiltonian becomes

$$\hat{\mathrm{H}}_{\mathrm{EM}} = \hbar\mathcal{H}_{\mathrm{EM}}(p=0) = \frac{1}{2\mathrm{L}}\hat{\pi}_b^2 - \frac{1}{\mathrm{L}}\hbar^2\sum_{p>0}\frac{d}{d\alpha_p}\frac{d}{d\alpha_{-p}}.$$

The total quantum Hamiltonian is

$$\hat{\mathrm{H}}_{\mathrm{R}} = \hat{\mathrm{H}}_0 + \hat{\mathrm{H}}_{\mathrm{EM}} - eb\hat{\mathrm{Q}}_{\mathrm{R,N}} - \frac{1}{2}\hbar\xi_{\mathrm{R}}.$$

If we multiply two operators that are finite linear combinations of the fermionic creation and annihilation operators, the ζ–function regulated operator product agrees with the naive product. However, if the operators involve infinite summations their naive product is not generally well defined. We then define the operator product by mutiplying the regulated operators with s large and positive and analytically continue the result to $s=0$. In this way we obtain the following relations (see Appendix)

$$[\rho(m),\rho(n)]_- = m\delta_{m,-n}, \tag{18}$$

$$[\hat{\mathrm{H}}_0,\rho(m)]_- = -\hbar\frac{2\pi}{\mathrm{L}}m\rho(m), \tag{19}$$

and

$$\begin{aligned} \frac{d}{db}\rho(m) &= 0,\\ \frac{d}{d\alpha_{\pm p}}\rho(m) &= \mp\frac{e\mathrm{L}}{2\pi}\delta_{p,\pm m}, \qquad (p>0). \end{aligned} \tag{20}$$

The quantum Gauss operator is

$$\hat{\mathrm{G}} = \hat{\mathrm{G}}_0 + \frac{2\pi}{\mathrm{L}^2}\sum_{p>0}\{\hat{\mathrm{G}}_+(p)e^{i\frac{2\pi}{\mathrm{L}}px} - \hat{\mathrm{G}}_-(p)e^{-i\frac{2\pi}{\mathrm{L}}px}\},$$

where

$$\hat{\mathrm{G}}_0 \equiv \frac{e}{\mathrm{L}}\hbar\rho(0),$$
$$\hat{\mathrm{G}}_{\pm}(p) \equiv \hbar(p\frac{d}{d\alpha_{\mp p}} \pm \frac{e\mathrm{L}}{2\pi}\rho(\pm p)).$$

Using 18 and 20, we easily get that $\rho(\pm p)$ are gauge-invariant:

$$[\hat{\mathrm{G}}_{+}(p), \rho(\pm q)]_{-} = 0,$$

$$[\hat{\mathrm{G}}_{-}(p), \rho(\pm q)]_{-} = 0,$$

$(p > 0, q > 0)$. The operators $\hat{\mathrm{G}}_{\pm}(p)$ don't commute with themselves,

$$[\hat{\mathrm{G}}_{+}(p), \hat{\mathrm{G}}_{-}(q)]_{-} = \hbar^2 \frac{e^2\mathrm{L}^2}{4\pi^2} p\delta_{p,q} \tag{21}$$

as well as with the Hamiltonian

$$[\hat{\mathrm{H}}_{\mathrm{R}}, \hat{\mathrm{G}}_{\pm}(p)]_{-} = \pm\hbar^3 \frac{e^2\mathrm{L}}{4\pi^2} \frac{d}{d\alpha_{\mp p}}. \tag{22}$$

The commutation relations 21 and 22 reflect an anomalous behaviour of the CSM.

3 QUANTUM CONSTRAINTS

3.1 QUANTUM SYMMETRY

In non-anomalous gauge theory, Gauss law is considered to be valid for physical states only. This identifies physical states as those which are gauge-invariant. The problem with the anomalous behaviour of the CSM, in terms of states in Hilbert space, is now apparent from Eqs. 21 – 22 : we cannot require that states be annihilated by the Gauss law generators $\hat{\mathrm{G}}_{\pm}(p)$.

Let us represent the action of the topologically trivial gauge transformations by the operator

$$\mathrm{U}_0(\tau) = \exp\{\frac{i}{\hbar}\hat{\mathrm{G}}_0\tau_0 + \frac{i}{\hbar}\sum_{p>0}(\hat{\mathrm{G}}_+\tau_+ + \hat{\mathrm{G}}_-\tau_-)\} \tag{23}$$

with τ_0 , $\tau_{\pm}(p)$ smooth, then

$$\begin{aligned} \mathrm{U}_0^{-1}(\tau)\alpha_{\pm p}\mathrm{U}_0(\tau) &= \alpha_{\pm} - ip\tau_{\mp}(p), \\ \mathrm{U}_0^{-1}(\tau)\frac{d}{d\alpha_{\pm p}}\mathrm{U}_0(\tau) &= \frac{d}{d\alpha_{\pm p}} + i(\frac{e\mathrm{L}}{2\pi})^2\tau_{\pm}(p). \end{aligned}$$

We find from Eq. 21 that

$$\mathrm{U}_0(\tau^{(1)})\mathrm{U}_0(\tau^{(2)}) = \exp\{2\pi i\omega_2(\tau^{(1)},\tau^{(2)})\}\mathrm{U}_0(\tau^{(1)}+\tau^{(2)}), \tag{24}$$

where

$$\omega_2(\tau^{(1)},\tau^{(2)}) \equiv -\frac{i}{4\pi}(\frac{e\mathrm{L}}{2\pi})^2\sum_{p>0}p(\tau_-^{(1)}\tau_+^{(2)} - \tau_+^{(1)}\tau_-^{(2)})$$

is a two-cocycle of the gauge group algebra. We are thus dealing with a projective representation.

The two-cocycle $\omega_2(\tau^{(1)},\tau^{(2)})$ is trivial, since it can be removed from 24 by a simple redefinition of $\mathrm{U}_0(\tau)$. Indeed, the modified operators

$$\tilde{\mathrm{U}}_0(\tau) = \exp\{i2\pi\alpha_1(\gamma;\tau)\}\cdot\mathrm{U}_0(\tau), \tag{25}$$

where

$$\alpha_1(\gamma,\tau) \equiv -\frac{1}{4\pi}(\frac{e\mathrm{L}}{2\pi})^2\sum_{p>0}(\alpha_{-p}\tau_- - \alpha_p\tau_+)$$

is a one-cocycle, satisfy the ordinary composition law

$$\tilde{\mathrm{U}}_0(\tau^{(1)})\tilde{\mathrm{U}}_0(\tau^{(2)}) = \tilde{\mathrm{U}}_0(\tau^{(1)}+\tau^{(2)}),$$

i.e. the action of the topologically trivial gauge transformations represented by 25 is unitary.

The modified Gauss law generators corresponding to 25 are

$$\tilde{\hat{\mathrm{G}}}_{\pm}(p) = \hat{\mathrm{G}}_{\pm}(p) \pm \hbar\frac{e^2\mathrm{L}^2}{8\pi^2}\alpha_{\pm p}. \tag{26}$$

The generators $\hat{\hat{\mathrm{G}}}_{\pm}(p)$ commute:

$$[\hat{\hat{\mathrm{G}}}_{+}(p), \hat{\hat{\mathrm{G}}}_{-}(q)]_{-} = 0.$$

This means that Gauss law can be maintained at the quantum level. We define physical states as those which are annihilated by $\hat{\hat{\mathrm{G}}}_{\pm}(p)$ [11] :

$$\hat{\hat{\mathrm{G}}}_{\pm}(p)|\mathrm{phys}; A\rangle = 0. \tag{27}$$

The zero component $\hat{\mathrm{G}}_0$ is a quantum generator of the global gauge transformations of the right-handed fermionic field, so the other quantum constraint is

$$\hat{\mathrm{Q}}_{\mathrm{R,N}}|\mathrm{phys}; A\rangle = 0. \tag{28}$$

3.2 ADIABATIC APPROACH

Let us show now that we can come to the quantum constraints 27 and 28 in a different way, using the adiabatic approximation [24, 25]. In the adiabatic approach, the dynamical variables are divided into two sets, one which we call fast variables and the other which we call slow variables. In our case, we treat the fermions as fast variables and the gauge fields as slow variables.

Let $\mathcal{A}^1$ be a manifold of all static gauge field configurations $A_1(x)$. On $\mathcal{A}^1$ a time-dependent gauge field $A_1(x,t)$ corresponds to a path and a periodic gauge field to a closed loop.

We consider the fermionic part of the second-quantized Hamiltonian $:\hat{\mathcal{H}}_{\mathrm{F}}:$ which depends on t through the background gauge field A_1 and so changes very slowly with time. We consider next the periodic gauge field $A_1(x,t)(0 \leq t < T)$. After a time T the periodic field $A_1(x,t)$ returns to its original value: $A_1(x,0) = A_1(x,T)$, so that $:\hat{\mathrm{H}}_{\mathrm{F}}:(0) =:\hat{\mathrm{H}}_{\mathrm{F}}:(T)$.

At each instant t we define eigenstates for $:\hat{\mathrm{H}}_{\mathrm{F}}:(t)$ by

$$:\hat{\mathrm{H}}_{\mathrm{F}}:(t)|\mathrm{F}, A(t)\rangle = \varepsilon_{\mathrm{F}}(t)|\mathrm{F}, A(t)\rangle.$$

The state $|\mathrm{F}=0, A(t)\rangle \equiv |\mathrm{vac}, A(t)\rangle$ is a ground state of $:\hat{\mathrm{H}}_{\mathrm{F}}:(t)$:

$$:\hat{\mathrm{H}}_{\mathrm{F}}:(t)|\mathrm{vac}, A(t)\rangle = 0.$$

The Fock states $|\mathrm{F}, A(t)\rangle$ depend on t only through their implicit dependence on A_1. They are assumed to be orthonormalized,

$$\langle \mathrm{F}', A(t)|\mathrm{F}, A(t)\rangle = \delta_{\mathrm{F},\mathrm{F}'},$$

and nondegenerate.

The time evolution of the wave function of our system (fermions in a background gauge field) is clearly governed by the Schrodinger equation:

$$i\hbar\frac{\partial\psi(t)}{\partial t} =: \hat{\mathrm{H}}_{\mathrm{F}} : (t)\psi(t).$$

For each t, this wave function can be expanded in terms of the "instantaneous" eigenstates $|\mathrm{F}, A(t)\rangle$.

Let us choose $\psi_{\mathrm{F}}(0) = |\mathrm{F}, A(0)\rangle$, i.e. the system is initially described by the eigenstate $|\mathrm{F}, A(0)\rangle$. According to the adiabatic approximation, if at $t = 0$ our system starts in an stationary state $|\mathrm{F}, A(0)\rangle$ of $: \hat{\mathrm{H}}_{\mathrm{F}} : (0)$, then it will remain, at any other instant of time t, in the corresponding eigenstate $|\mathrm{F}, A(t)\rangle$ of the instantaneous Hamiltonian $: \hat{\mathrm{H}}_{\mathrm{F}} : (t)$. In other words, in the adiabatic approximation transitions to other eigenstates are neglected.

Thus, at some time t later our system will be described up to a phase by the same Fock state $|\mathrm{F}, A(t)\rangle$:

$$\psi_{\mathrm{F}}(t) = \mathrm{C}_{\mathrm{F}}(t) \cdot |\mathrm{F}, A(t)\rangle, \tag{29}$$

where $\mathrm{C}_{\mathrm{F}}(t)$ is yet undetermined phase.

To find this phase, we insert 29 into the Schrodinger equation :

$$\hbar\dot{\mathrm{C}}_{\mathrm{F}}(t) = -i\mathrm{C}_{\mathrm{F}}(t)\varepsilon_{\mathrm{F}}(t) - \hbar\mathrm{C}_{\mathrm{F}}(t)\langle\mathrm{F}, A(t)|\frac{\partial}{\partial t}|\mathrm{F}, A(t)\rangle. \tag{30}$$

Solving 30, we get

$$\mathrm{C}_{\mathrm{F}}(t) = \exp\{-\frac{i}{\hbar}\int_0^t dt'\varepsilon_{\mathrm{F}}(t') - \int_0^t dt'\langle\mathrm{F}, A(t')|\frac{\partial}{\partial t'}|\mathrm{F}, A(t')\rangle\}.$$

For $t = T$, $|\mathrm{F}, A(T)\rangle = |\mathrm{F}, A(0)\rangle$ (the instantaneous eigenfunctions are chosen to be periodic in time) and

$$\psi_{\mathrm{F}}(T) = \exp\{i\gamma_{\mathrm{F}}^{\mathrm{dyn}} + i\gamma_{\mathrm{F}}^{\mathrm{Berry}}\} \cdot \psi_{\mathrm{F}}(0),$$

where

$$\gamma_{\mathrm{F}}^{\mathrm{dyn}} \equiv -\frac{1}{\hbar}\int_0^T dt \cdot \varepsilon_{\mathrm{F}}(t),$$

while

$$\gamma_{\mathrm{F}}^{\mathrm{Berry}} \equiv \int_0^T dt \int_{-\mathrm{L}/2}^{\mathrm{L}/2} dx\dot{A}_1(x,t)\langle\mathrm{F}, A(t)|i\frac{\delta}{\delta A_1(x,t)}|\mathrm{F}, A(t)\rangle \tag{31}$$

is Berry's phase [25].

If we define the $U(1)$ connection

$$\mathcal{A}_{\mathrm{F}}(x,t) \equiv \langle\mathrm{F}, A(t)|i\frac{\delta}{\delta A_1(x,t)}|\mathrm{F}, A(t)\rangle, \tag{32}$$

then

$$\gamma_{\mathrm{F}}^{\mathrm{Berry}} = \int_0^T dt \int_{-\mathrm{L}/2}^{\mathrm{L}/2} dx\dot{A}_1(x,t)\mathcal{A}_{\mathrm{F}}(x,t).$$

We see that upon parallel transport around a closed loop on $\mathcal{A}^1$ the Fock state $|\mathrm{F}, A(t)\rangle$ acquires an additional phase which is integrated exponential of $\mathcal{A}_{\mathrm{F}}(x,t)$. Whereas the dynamical phase $\gamma_{\mathrm{F}}^{\mathrm{dyn}}$ provides information about the duration of the evolution, the Berry's phase reflects the nontrivial holonomy of the Fock states on $\mathcal{A}^1$.

However, a direct computation of the diagonal matrix elements of $\frac{\delta}{\delta A_1(x,t)}$ in 31 requires a globally single-valued basis for the eigenstates $|\mathrm{F}, A(t)\rangle$ which is not available. The connection 32 can be defined only locally on $\mathcal{A}^1$, in regions where $[\frac{eb\mathrm{L}}{2\pi}]$ is fixed. The values of A_1 in regions of different $[\frac{eb\mathrm{L}}{2\pi}]$ are connected by topologically nontrivial gauge transformations. If $[\frac{eb\mathrm{L}}{2\pi}]$ changes, then there is a nontrivial spectral flow , i.e. some of energy levels of the first quantized fermionic Hamiltonian cross zero and change sign. This means that the definition of the Fock vacuum of the second quantized fermionic Hamiltonian changes (see Eq. 8). Since the creation and annihilation operators $a^\dagger, a$ are continuous functionals of $A_1(x)$, the definition of all excited Fock states $|\mathrm{F}, A(t)\rangle$ is also discontinuous. The connection $\mathcal{A}_{\mathrm{F}}$ is not therefore well-defined globally. Its global characterization necessiates the usual introduction of transition functions.

Furthermore, $\mathcal{A}_{\mathrm{F}}$ is not invariant under A–dependent redefinitions of the phases of the Fock states: $|\mathrm{F}, A(t)\rangle \rightarrow \exp\{-i\chi[A]\}|\mathrm{F}, A(t)\rangle$, and transforms like a $U(1)$ vector potential

$$\mathcal{A}_{\mathrm{F}} \rightarrow \mathcal{A}_{\mathrm{F}} + \frac{\delta\chi[A]}{\delta A_1}.$$

For these reasons, to calculate $\gamma_{\mathrm{F}}^{\mathrm{Berry}}$ it is more convenient to compute first the $U(1)$ curvature tensor

$$\mathcal{F}_{\mathrm{F}}(x,y,t) \equiv \frac{\delta}{\delta A_1(x,t)} \mathcal{A}_{\mathrm{F}}(y,t) - \frac{\delta}{\delta A_1(y,t)} \mathcal{A}_{\mathrm{F}}(x,t) \tag{33}$$

and then deduce $\mathcal{A}_{\mathrm{F}}$.

For simplicity, let us compute the vacuum curvature tensor $\mathcal{F}_{\mathrm{F}=0}(x,y,t)$. Substituting 32 into 33, we get

$$\begin{aligned}\mathcal{F}_{\mathrm{F}=0}(x,y,t) = i \sum_{\mathrm{F}\neq 0} \{ &\langle \mathrm{vac}, A(t)| \frac{\delta}{\delta A_1(y,t)} |\mathrm{F}, A(t)\rangle \langle \mathrm{F}, A(t)| \frac{\delta}{\delta A_1(x,t)} |\mathrm{vac}, A(t)\rangle \\ &- (x \longleftrightarrow y)\},\end{aligned} \tag{34}$$

where the summation is over the complete set of states $|\mathrm{F}, A(t)\rangle$.

Using the formula

$$\langle \mathrm{vac}, A(t)| \frac{\delta}{\delta A_1(x,t)} |\mathrm{F}, A(t)\rangle = \frac{1}{\varepsilon_{\mathrm{F}}} \langle \mathrm{vac}, A(t)| \frac{\delta : \hat{\mathrm{H}}_{\mathrm{F}} : (t)}{\delta A_1(x,t)} |\mathrm{F}, A(t)\rangle,$$

we rewrite 34 as

$$\mathcal{F}_{\mathrm{F}=0}(x,y,t) = i \sum_{\mathrm{F}\neq 0} \frac{1}{\varepsilon_{\mathrm{F}}^2} \{ \langle \mathrm{vac}, A(t)| \frac{\delta : \hat{\mathrm{H}}_{\mathrm{F}} : (t)}{\delta A_1(y,t)} |\mathrm{F}, A(t)\rangle \cdot \langle \mathrm{F}, A(t)| \frac{\delta : \hat{\mathrm{H}}_{\mathrm{F}} : (t)}{\delta A_1(x,t)} |\mathrm{vac}, A(t)\rangle$$

$$- (x \longleftrightarrow y)\}. \tag{35}$$

Since $\frac{\delta :\hat{\rm H}_{\rm F}:(t)}{\delta A_1}$ is quadratic in $a^\dagger, a$, only excited states of the type

$$|{\rm F}, A(t)\rangle \longleftrightarrow a_m^\dagger a_n |{\rm vac}, A(t)\rangle \qquad (n \leq [\frac{eb{\rm L}}{2\pi}], m > [\frac{eb{\rm L}}{2\pi}])$$

with $\varepsilon_{\rm F} = \frac{2\pi}{\rm L}\hbar(m-n)$ contribute to 35 which takes then the form

$$\mathcal{F}_{\rm F=0}(x,y,t) = i\frac{{\rm L}^2}{4\pi^2}\sum_{m\neq n}\frac{1}{\hbar^2(m-n)^2}\{\langle {\rm vac}, A(t)|\frac{\delta : \hat{\rm H}_{\rm F} : (t)}{\delta A_1(y,t)} a_m^\dagger a_n |{\rm vac}, A(t)\rangle \cdot$$

$$\langle {\rm vac}, A(t)| a_n^\dagger a_m \frac{\delta : \hat{\rm H}_{\rm F} : (t)}{\delta A_1(x,t)}|{\rm vac}, A(t)\rangle - (x \longleftrightarrow y)\}. \tag{36}$$

With $: \hat{\rm H}_{\rm F} : (t)$ given by 15, Eq. 36 is evaluated as

$$\mathcal{F}_{\rm F=0} = \frac{e^2}{2\pi}\sum_{n>0}\frac{1}{n}\sin(\frac{2\pi}{\rm L}n(x-y)) = \frac{e^2}{4\pi}\epsilon(x-y) - \frac{e^2}{2\pi{\rm L}}(x-y). \tag{37}$$

The corresponding $U(1)$ connection is easily deduced as

$$\mathcal{A}_{\rm F=0}(x,t) = -\frac{1}{2}\int_{-{\rm L}/2}^{{\rm L}/2} dy \mathcal{F}_{\rm F=0}(x,y,t) A_1(y,t).$$

The Berry phase becomes

$$\gamma_{\rm F=0}^{\rm Berry} = -\frac{1}{2}\int_0^T dt \int_{-{\rm L}/2}^{{\rm L}/2} dx \int_{-{\rm L}/2}^{{\rm L}/2} dy \dot{A}_1(x,t)\mathcal{F}_{\rm F=0}(x,y,t)A_1(y,t).$$

We see that in the limit ${\rm L} \to \infty$, when the second term in 37 may be neglected, the $U(1)$ curvature tensor coincides with that obtained in [14, 15], while the Berry phase is

$$\gamma_{\rm F=0}^{\rm Berry} = \int_0^T dt \int_{-\infty}^{\infty} dx \mathcal{L}(x,t),$$

where

$$\mathcal{L}(x,t) \equiv -\frac{e^2}{8\pi^2}\int_{-\infty}^{\infty} dy \dot{A}_1(x,t)\epsilon(x-y)A_1(y,t)$$

is a non-local part of the effective Lagrange density of the CSM [12].

In terms of the Fourier components, the connection $\mathcal{A}_{\rm F=0}$ is rewritten as

$$\begin{aligned}
\langle {\rm vac}, A(t)|\frac{d}{db(t)}|{\rm vac}, A(t)\rangle &= 0, \\
\langle {\rm vac}, A(t)|\frac{d}{d\alpha_{\pm p}(t)}|{\rm vac}, A(t)\rangle &\equiv \mathcal{A}_{\pm}(p,t) = \pm\frac{e^2{\rm L}^2}{8\pi^2}\frac{1}{p}\alpha_{\mp p},
\end{aligned}$$

so the nonvanishing curvature is

$$\mathcal{F}_{+-} \equiv \frac{d}{d\alpha_{-p}}\mathcal{A}_+ - \frac{d}{d\alpha_p}\mathcal{A}_- = \frac{e^2\mathrm{L}^2}{4\pi^2}\frac{1}{p}.$$

A parallel transportation of the vacuum $|\mathrm{vac}, A(t)\rangle$ around a closed loop in (α_p, α_{-p}) – space $(p > 0)$ yields back the same vacuum state multiplied by the phase

$$\gamma^{\mathrm{Berry}}_{\mathrm{F}=0} = \frac{e^2\mathrm{L}^2}{4\pi^2}\int_0^T dt \sum_{p>0}\frac{1}{p} i\alpha_p\dot{\alpha}_{-p}.$$

However, this phase associated with the projective representation of the gauge group is trivial, since it can be removed. If we redefine the momentum operators as

$$\frac{d}{d\alpha_{\pm p}} \longrightarrow \frac{\tilde{d}}{d\alpha_{\pm p}} \equiv \frac{d}{d\alpha_{\pm p}} \mp \frac{e^2\mathrm{L}^2}{8\pi^2}\frac{1}{p}\alpha_{\mp p}, \tag{38}$$

then the corresponding connection and curvature vanish:

$$\begin{aligned}
\tilde{\mathcal{A}}_\pm &\equiv \langle \mathrm{vac}, A(t)|\frac{\tilde{d}}{d\alpha_{\pm p}}|\mathrm{vac}, A(t)\rangle = 0, \\
\tilde{\mathcal{F}}_{+-} &= \frac{\tilde{d}}{d\alpha_{-p}}\tilde{\mathcal{A}}_+ - \frac{\tilde{d}}{d\alpha_p}\tilde{\mathcal{A}}_- = 0.
\end{aligned}$$

The modified momentum operators are noncommuting:

$$[\frac{\tilde{d}}{d\alpha_p}, \frac{\tilde{d}}{d\alpha_{-q}}]_- = \frac{e^2\mathrm{L}^2}{4\pi^2}\frac{1}{p}\delta_{p,q}.$$

Following 38, we modify the Gauss law generators as

$$\hat{\mathrm{G}}_\pm(p) \longrightarrow \hat{\tilde{\mathrm{G}}}_\pm(p) = \hbar(\frac{\tilde{d}}{d\alpha_{\mp p}} \pm \frac{e\mathrm{L}}{2\pi}\rho(\pm p))$$

that coincides with 26. The modified Gauss law generators have vanishing vacuum expectation values,

$$\langle \mathrm{vac}, A(t)|\hat{\tilde{\mathrm{G}}}_\pm(p,t)|\mathrm{vac}, A(t)\rangle = 0.$$

This justifies the definition 27.

For the zero component $\hat{\mathrm{G}}_0$, the vacuum expectation value

$$\langle \mathrm{vac}, A(t)|\hat{\mathrm{G}}_0|\mathrm{vac}, A(t)\rangle = -\hbar\frac{e}{2}\eta_{\mathrm{R}}$$

can be also made equal to zero by the redefinition

$$\hat{\mathrm{G}}_0 \longrightarrow \hat{\tilde{\mathrm{G}}}_0 + \hbar\frac{e}{2}\eta_{\mathrm{R}} = \frac{e}{\mathrm{L}}\hbar : \rho(0) :$$

that leads to 28.

Thus, both quantum constraints 27 and 28 can be realized in the framework of the adiabatic approximation.

4 PHYSICAL QUANTUM CSM

4.1 CONSTRUCTION OF PHYSICAL HAMILTONIAN

1. From the point of view of Dirac quantization, there are many physically equivalent classical theories of a system with first-class constraints. As mentioned, the origin of such an ambiguity lies in a gauge freedom. For the classical CSM, the gauge freedom is characterized by an arbitrary $v_{\rm H}(x)$ in 4. If we use the Fourier expansion for $v_{\rm H}(x)$, then the general form of the classical Hamiltonian is rewritten as

$$\tilde{\rm H}_{\rm R} = {\rm H}_{\rm R} + \sum_{p>0} (v_{{\rm H},+}{\rm G}_+ + v_{{\rm H},-}{\rm G}_-). \tag{39}$$

Any Hamiltonian $\tilde{\rm H}_{\rm R}$ with fixed nonzero $(v_{{\rm H},-}, v_{{\rm H},+})$ gives rise to the same weak equations of motion as those deduced from ${\rm H}_{\rm R}$, although the strong form of these equations may be quite different. The physics is however described by the weak equations. Different $(v_{{\rm H},-}, v_{{\rm H},+})$ lead to different mathematical descriptions of the same physical situation.

To construct the quantum theory of any system with first-class constraints, we usually quantize one of the corresponding classical theories. All the possible quantum theories constructed in this way are believed to be equivalent to each other.

In the case, when gauge degrees of freedom are anomalous, the situation is different: the physical equivalence of quantum Hamiltonians is lost. For the CSM, the quantum Hamiltonian $\hat{\tilde{\rm H}}_{\rm R}$ does not reduce to $\hat{\rm H}_{\rm R}$ on the physical states:

$$\hat{\tilde{\rm H}}_{\rm R}|{\rm phys}; A\rangle \neq \hat{\rm H}_{\rm R}|{\rm phys}; A\rangle.$$

The quantum theory consistently describing the dynamics of the CSM should be definitely compatible with 27. The corresponding quantum Hamiltonian is then defined by the conditions

$$[\hat{\tilde{\rm H}}_{\rm R}, \hat{\tilde{\rm G}}_{\pm}(p)]_- = 0 \qquad (p > 0) \tag{40}$$

which specify that $\hat{\tilde{\rm H}}_{\rm R}$ must be invariant under the modified topologically trivial gauge transformations generated by $\hat{\tilde{\rm G}}_{\pm}(p)$.

The conditions 40 can be considered as a system of equations for the Lagrange multipliers $\hat{v}_{{\rm H},\pm}$ which become operators at the quantum level. These equations are

$$\sum_{q>0} \{\hat{\rm G}_+(q)[\hat{v}_{{\rm H},+}(q), \hat{\tilde{\rm G}}_{\pm}(p)]_- + \hat{\rm G}_-(q)[\hat{v}_{{\rm H},-}(q), \hat{\tilde{\rm G}}_{\pm}(p)]_-\} \mp \hbar^2 \frac{e^2{\rm L}^2}{8\pi^2} p\hat{v}_{{\rm H},\mp}(p)$$

$$= \mp\hbar^3 \frac{e^2{\rm L}}{8\pi^2} \frac{d}{d\alpha_{\mp p}}$$

and fix $\hat{v}_{{\rm H},\pm}(p)$ in the form

$$\hat{v}_{{\rm H},\pm}(p) = \pm\frac{e}{2\pi}\hbar\frac{1}{p^2}(\rho(\mp p) + \frac{e{\rm L}}{8\pi}\alpha_{\mp p}).$$

Substituting $\hat{v}_{\mathrm{H},\pm}(p)$ into the quantum counterpart of 39, on the physical states $|\mathrm{phys}; A\rangle$ we get

$$\frac{1}{2}\sum_{p>0}\{[\hat{v}_{\mathrm{H},+}(p), \hat{\mathrm{G}}_+(p)]_+ + [\hat{v}_{\mathrm{H},-}(p), \hat{\mathrm{G}}_-(p)]_+\} = \frac{1}{\mathrm{L}^2}\hbar^2\sum_{p>0}(\frac{d}{d\alpha_p}\frac{d}{d\alpha_{-p}} - \frac{1}{2}[\frac{\tilde{d}}{d\alpha_p}, \frac{\tilde{d}}{d\alpha_{-p}}]_+), \tag{41}$$

i.e. the last term in the right-hand side of 39 contributes only to the electromagnetic part of the Hamiltonian, changing $\frac{d}{d\alpha_\pm}$ by $\frac{\tilde{d}}{d\alpha_\pm}$:

$$\hat{\mathrm{H}}_{\mathrm{EM}} \rightarrow \frac{1}{2\mathrm{L}}\hat{\pi}_b^2 - \frac{1}{2\mathrm{L}}\hbar^2\sum_{p>0}[\frac{\tilde{d}}{d\alpha_p}, \frac{\tilde{d}}{d\alpha_{-p}}]_+.$$

2. The topologically nontrivial gauge transformations change the integer part of $\frac{eb\mathrm{L}}{2\pi}$:

$$\begin{aligned}
[\frac{eb\mathrm{L}}{2\pi}] &\rightarrow [\frac{eb\mathrm{L}}{2\pi}] + n, \\
\{\frac{eb\mathrm{L}}{2\pi}\} &\rightarrow \{\frac{eb\mathrm{L}}{2\pi}\}, \\
\hat{\psi}_{\mathrm{R}} &\rightarrow \exp\{i\frac{2\pi n}{\mathrm{L}}x\}\hat{\psi}_{\mathrm{R}}.
\end{aligned}$$

The action of the topologically nontrivial gauge transformations on the states can be represented by the operators

$$\mathrm{U}_n = \exp\{-\frac{i}{\hbar}n \cdot \hat{\mathrm{T}}_b\} \cdot \mathrm{U}_0 \tag{42}$$

where

$$\hat{\mathrm{T}}_b \equiv \hat{\pi}_{[\frac{eb\mathrm{L}}{2\pi}]} - \frac{2\pi}{\mathrm{L}}\int_{-\mathrm{L}/2}^{\mathrm{L}/2} dxx \cdot \hat{j}_{\mathrm{R}}(x) \equiv -i\hbar\frac{d}{d[\frac{eb\mathrm{L}}{2\pi}]} + i\hbar\sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}}\frac{(-1)^p}{2p}\rho(p)$$

and U_0 is given by 23.

To identify the gauge transformation as belonging to the nth topological class we use the index n in 42. The case $n = 0$ corresponds to the topologically trivial gauge transformations.

The Fourier components of the fermionic current are transformed as

$$\rho(\pm p) \rightarrow \rho(\pm p) - (-1)^p \cdot n, \qquad (p > 0).$$

The composition law 24 is valid for the topologically nontrivial gauge transformations, too. The modified topologically nontrivial gauge transformation operators are

$$\tilde{\mathrm{U}}_n = \exp\{-\frac{i}{\hbar}n \cdot \hat{\mathrm{T}}_b\} \cdot \tilde{\mathrm{U}}_0.$$

On the physical states

$$\tilde{\mathrm{U}}_n|\mathrm{phys}; A\rangle = (\exp\{-\frac{i}{\hbar}\hat{\mathrm{T}}_b\})^n|\mathrm{phys}; A\rangle.$$

Let $|\mathrm{phys}; A; n\rangle$ be a physical state in which the integer part of $\frac{eb\mathrm{L}}{2\pi}$ is equal n. Then in the state $\exp\{-\frac{i}{\hbar}\hat{\mathrm{T}}_b\}|\mathrm{phys}; A; n\rangle = |\mathrm{phys}; A; n+1\rangle$ the integer part of $\frac{eb\mathrm{L}}{2\pi}$ is equal $n+1$, i.e. the topologically nontrivial gauge transformation operator $\exp\{-\frac{i}{\hbar}\hat{\mathrm{T}}_b\}$ increases $[\frac{eb\mathrm{L}}{2\pi}]$ by one. The operator $\exp\{\frac{i}{\hbar}\hat{\mathrm{T}}_b\}$ decreases $[\frac{eb\mathrm{L}}{2\pi}]$ by one.

The vacuum state $|\mathrm{vac}; A; n\rangle$ is defined as follows

$$\begin{aligned} a_m|\mathrm{vac}; A; n\rangle &= 0 \quad \text{for} \quad m \geq n+1, \\ a_m^\dagger|\mathrm{vac}; A; n\rangle &= 0 \quad \text{for} \quad m < n+1, \end{aligned} \tag{43}$$

the levels with energy lower than $\varepsilon_{\mathrm{R,n+1}}$ being filled and the others being empty. While the vacuum 8 is defined such that it is always the lowest energy state at any configuration of the gauge field, the vacuum 43 is the lowest energy state only when the global gauge field degree of freedom b satisfies the condition $n \leq \frac{eb\mathrm{L}}{2\pi} \leq n+1$, i.e. $[\frac{eb\mathrm{L}}{2\pi}] = n$.

Among all states $|\mathrm{phys}; A\rangle$ one may identify the eigenstates of the operators of the physical variables. The action of the topologically nontrivial gauge transformations on such states may, generally speaking, change only the phase of these states by a C–number, since with any gauge transformations both topologically trivial and nontrivial, the operators of the physical variables and the observables cannot be changed. Using $|\mathrm{phys}; \theta\rangle$ to designate these physical states, we have

$$\exp\{\mp\frac{i}{\hbar}\hat{\mathrm{T}}_b\}|\mathrm{phys}; \theta\rangle = e^{\pm i\theta}|\mathrm{phys}; \theta\rangle, \tag{44}$$

The states $|\mathrm{phys}; \theta\rangle$ obeying 44 are easily constructed in the form

$$|\mathrm{phys}; \theta\rangle = \sum_{n \in \mathcal{Z}} e^{-in\theta}(\exp\{-\frac{i}{\hbar}\hat{\mathrm{T}}_b\})^n|\mathrm{phys}; A\rangle \tag{45}$$

(so called θ–states [26, 27]), where $|\mathrm{phys}; A\rangle$ is an arbitrary physical state from 27.

In one dimension the θ–parameter is related to a constant background electric field . To show this, we introduce states which are invariant even against the topologically nontrivial gauge transformations. Recalling that $[\frac{eb\mathrm{L}}{2\pi}]$ is shifted by n under a gauge transformation from the nth topological class, we easily construct such states as

$$|\mathrm{phys}\rangle \equiv \exp\{i[\frac{eb\mathrm{L}}{2\pi}]\theta\}|\mathrm{phys}; \theta\rangle. \tag{46}$$

The states $|\mathrm{phys}\rangle$ continue to be annihilated by $\hat{\tilde{\mathrm{G}}}_\pm(p)$, and are also invariant under the topologically nontrivial gauge transformations, so we can require that

$$\hat{\mathrm{T}}_b|\mathrm{phys}\rangle = 0. \tag{47}$$

On the states 46 the electromagnetic part of the Hamiltonian takes the form

$$\hat{\mathrm{H}}_{\mathrm{EM}} \to \frac{1}{2\mathrm{L}}(\hat{\pi}_b + \hbar\frac{\mathrm{L}}{2}\mathcal{E}_\theta)^2 - \frac{1}{2\mathrm{L}}\hbar^2 \sum_{p>0}[\frac{\tilde{d}}{d\alpha_p}, \frac{\tilde{d}}{d\alpha_{-p}}]_+,$$

i.e. the momentum $\hat{\pi}_b$ is supplemented by the electric field strength $\mathcal{E}_\theta \equiv \frac{e}{\pi}\theta$.

The quantum Hamiltonian invariant under the topologically trivial gauge transformations is not unique. We can add to it any linear combination of the gauge-invariant operators $\rho(\pm p)$:

$$\hat{\hat{\mathrm{H}}} \to \hat{\hat{\mathrm{H}}} + \beta_{\mathrm{H},0} + \sum_{p>0}(\beta_{\mathrm{H},+}(p)\rho(p) + \beta_{\mathrm{H},-}(p)\rho(-p))$$

where $\beta_{\mathrm{H},0}$, $\beta_{\mathrm{H},\pm}$ are yet undetermined functions. The conditions 40 does not clearly fix these functions.

The Hamiltonian of the consistent quantum theory of the CSM should be invariant under the topologically nontrivial gauge transformations as well. So next to 40 is the following condition

$$[\hat{\hat{\mathrm{H}}}_{\mathrm{R}}, \hat{\mathrm{T}}_b]_- = 0. \tag{48}$$

The condition 48 can be then rewritten as a system of three equations for $(\beta_{\mathrm{H},0}, \beta_{\mathrm{H},\pm})$ and is solved up to constants independent of $[\frac{eb\mathrm{L}}{2\pi}]$ by

$$\begin{aligned} \beta^s_{\mathrm{H},0} &= \hbar([\frac{eb\mathrm{L}}{2\pi}])^2 \sum_{p>0} \frac{1}{p}\varepsilon^s_{\mathrm{R}}(p), \\ \beta^s_{\mathrm{H},\pm} &= \hbar[\frac{eb\mathrm{L}}{2\pi}]\frac{(-1)^p}{p}\varepsilon^s_{\mathrm{R}}(p), \end{aligned} \tag{49}$$

where

$$\varepsilon^s_{\mathrm{R}}(p) \equiv \frac{2\pi}{\mathrm{L}}p|\lambda\varepsilon_{p,\mathrm{R}}|^{-s} + \frac{e^2\mathrm{L}}{4\pi^2}\frac{1}{p}\hbar.$$

3. If we apply the bosonization procedure, then the bosonized version of the regularized free fermionic Hamiltonian is (see Appendix)

$$\hat{\mathrm{H}}^s_0 = \frac{2\pi}{\mathrm{L}}\hbar \sum_{p>0} |\lambda\varepsilon_{p,\mathrm{R}}|^{-s}\rho_s(-p)\rho_s(p)$$

With 41 and 49, on the physical states we then get

$$\hat{\hat{\mathrm{H}}}_{\mathrm{R}}|\mathrm{phys}; A\rangle = \hat{\mathrm{H}}_{\mathrm{phys}}|\mathrm{phys}; A\rangle$$

where

$$\begin{aligned} \hat{\mathrm{H}}^s_{\mathrm{phys}} = \frac{\hbar}{2}\sum_{p>0}\frac{1}{p}\varepsilon^s_{\mathrm{R}}(p)\rho_s(-p)\rho_s(p) + \hbar[\frac{eb\mathrm{L}}{2\pi}]\sum_{p>0}\frac{(-1)^p}{p}\varepsilon^s_{\mathrm{R}}(p)\rho_s(p) \\ + \frac{1}{2\mathrm{L}^2}\int_{-\mathrm{L}/2}^{\mathrm{L}/2} dx \cdot (\hat{\hat{\pi}}^s_b(x))^2 - \frac{1}{2}\xi_{\mathrm{R}}\hbar. \end{aligned}$$

We have defined here the modified momentum operator

$$\hat{\hat{\pi}}^s_b(x) = \hat{\pi}_b - \frac{\mathrm{L}}{2}\hbar\mathcal{E}_s(x)$$

where

$$\mathcal{E}_s(x) \equiv -\frac{2e_s x}{\mathrm{L}}[\frac{eb\mathrm{L}}{2\pi}]$$

and

$$e_s \equiv \sqrt{e^2 + \frac{48\pi}{\mathrm{L}^2}\frac{1}{\hbar}\sum_{p>o}|\lambda\varepsilon_{p,\mathrm{R}}|^{-s}}. \tag{50}$$

We see that the line integral b not only represents the physical degrees of freedom of the gauge field, but also creates on the circle $-\frac{\mathrm{L}}{2} \leq x \leq \frac{\mathrm{L}}{2}$ a background linearly rising electric field in which the physical degrees of freedom of the model are moving. On the states 46 the density of the electromagnetic part of the physical Hamiltonian is

$$\hat{\mathcal{H}}_{\mathrm{EM}}^{s,\mathrm{phys}} \rightarrow \frac{1}{2\mathrm{L}^2}(\hat{\pi}_b + \hbar\frac{\mathrm{L}}{2}(\mathcal{E}_\theta - \mathcal{E}_s))^2.$$

While the constant background electric field is general in one-dimensional gauge models defined on the circle, the linearly rising one is specific to the CSM [16].

For large L, we may neglect the second term in the parentheses in 50, so $e_s \simeq e$ and

$$\mathcal{E}_s(x) \simeq -\frac{2ex}{\mathrm{L}}[\frac{eb\mathrm{L}}{2\pi}]$$

that coincides with the expression given for the background electric field strength in [16]. If we evaluate e_s at large s and then take the limit $s \rightarrow 0$, we get again that

$$\lim_{s\rightarrow 0} e_s = e$$

and

$$\lim_{s\rightarrow 0}\mathcal{E}_s(x) = -\frac{2ex}{\mathrm{L}}[\frac{eb\mathrm{L}}{2\pi}].$$

The commutation relations for $\hat{\tilde{\pi}}_b$ are

$$[\hat{\tilde{\pi}}_b(x), \hat{\tilde{\pi}}_b(y)]_- = i\hbar^2\frac{e^2\mathrm{L}}{2\pi}(x-y).$$

The background linearly rising electric field may be described by the scalar potential

$$\varphi(x) = \frac{e}{\mathrm{L}}x^2[\frac{eb\mathrm{L}}{2\pi}]$$

and is created by the charge uniformly distributed on the circle with the density

$$\rho(x) = -\frac{2}{\mathrm{L}}[\frac{eb\mathrm{L}}{2\pi}].$$

The topologically nontrivial gauge transformations change $\rho(x)$ as follows

$$\rho(x) \rightarrow \rho(x) - \frac{2}{\mathrm{L}}n.$$

Only non-zero $[\frac{eb\mathrm{L}}{2\pi}]$'s correspond to the nonvanishing background charge density. Moreover, for non-zero $[\frac{eb\mathrm{L}}{2\pi}]$ the fermionic physical degrees of freedom and b are not decoupled in the physical Hamiltonian. Such decoupling for all values of $[\frac{eb\mathrm{L}}{2\pi}]$ is known to occur in the Schwinger model [21, 28]. It is just the background linearly rising electric field that couples b to the fermionic physical degrees of freedom.

We see also that the spectrum of the fermionic part of the physical Hamiltonian

$$\varepsilon_{\mathrm{R}}(p) = \lim_{s \to \infty} \varepsilon_{\mathrm{R}}^{s}(p) = \frac{\mathrm{L}}{2\pi p}[(\frac{2\pi p}{\mathrm{L}})^2 + \frac{e^2}{2\pi}\hbar]$$

is non-relativistic that indicates the breakdown of relativistic invariance.

4.2 EXOTIZATION

Let us present now the procedure which we call exotization. We can formally decouple the matter and gauge field degrees of freedom by introducing the exotic statistics matter field [17].

We define the composite field

$$\tilde{\psi}_{\mathrm{R}}(x) = \exp\{i\frac{\pi}{\mathrm{L}}x + \frac{i}{\hbar}\frac{2\pi}{e\mathrm{L}}\hat{\tilde{\pi}}_b(x)\} \cdot \psi_{\mathrm{R}}(x). \tag{51}$$

The field $\tilde{\psi}_{\mathrm{R}}(x)$ has the commutation relations

$$\begin{aligned} \tilde{\psi}_{\mathrm{R}}^{\star}(x)\tilde{\psi}_{\mathrm{R}}(y) \quad &+ \quad e^{+i\mathrm{F}(x,y)}\tilde{\psi}_{\mathrm{R}}(y)\tilde{\psi}_{\mathrm{R}}^{\star}(x) = \delta(x-y), \\ \tilde{\psi}_{\mathrm{R}}(x)\tilde{\psi}_{\mathrm{R}}(y) \quad &+ \quad e^{-i\mathrm{F}(x,y)}\tilde{\psi}_{\mathrm{R}}(y)\tilde{\psi}_{\mathrm{R}}(x) = 0, \end{aligned} \tag{52}$$

where $\mathrm{F}(x,y) \equiv \frac{2\pi}{\mathrm{L}}(x-y)$. The commutation relations 52 are indicative of an exotic statistics of $\tilde{\psi}_{\mathrm{R}}(x)$. This field is neither fermionic nor bosonic. Only for $x = y$ Eqs. 52 become anti-commutators: $\tilde{\psi}_{\mathrm{R}}(x)$ (and $\tilde{\psi}_{\mathrm{R}}^{\star}(x)$) anticommutes with itself, i.e. behaves as a fermionic field.

Using 6 and 51, we get the Fourier expansion for the exotic field $\tilde{\psi}_{\mathrm{R}}(x)$:

$$\tilde{\psi}_{\mathrm{R}}^{s}(x) = \sum_{n \in \mathcal{Z}} \tilde{a}_n \langle x|n;\mathrm{R}\rangle |\lambda\varepsilon_{n,\mathrm{R}}|^{-s/2}$$

where

$$\tilde{a}_n \equiv \exp\{\frac{i}{\hbar}\frac{2\pi}{e\mathrm{L}}\hat{\pi}_b\} a_{n-[\frac{eb\mathrm{L}}{2\pi}]},$$

$$\tilde{a}_n^{\dagger} \equiv a_{n-[\frac{eb\mathrm{L}}{2\pi}]}^{\dagger} \exp\{-\frac{i}{\hbar}\frac{2\pi}{e\mathrm{L}}\hat{\pi}_b\} \neq \tilde{a}_{-n}.$$

The exotic creation and annihilation operators $\tilde{a}_n, \tilde{a}_n^{\dagger}$ fulfil the following commutation relations algebra:

$$\begin{aligned} \tilde{a}_n^{\dagger}\tilde{a}_m + \tilde{a}_{m-1}\tilde{a}_{n-1}^{\dagger} &= \delta_{m,n}, \\ \tilde{a}_n\tilde{a}_m^{\dagger} + \tilde{a}_{m+1}^{\dagger}\tilde{a}_{n-1} &= \delta_{m,n}, \\ \tilde{a}_n\tilde{a}_m + \tilde{a}_{m+1}\tilde{a}_{n-1} &= 0, \\ \tilde{a}_n^{\dagger}\tilde{a}_m^{\dagger} + \tilde{a}_{m-1}^{\dagger}\tilde{a}_{n+1}^{\dagger} &= 0. \end{aligned}$$

Let us introduce the new Fock vacuum $\overline{|\mathrm{vac}; A\rangle}$ defined as

$$\tilde{a}_n \overline{|\mathrm{vac}; A\rangle} = 0 \quad \text{for} \quad n > [\frac{eb\mathrm{L}}{2\pi}],$$
$$\tilde{a}_n^\dagger \overline{|\mathrm{vac}; A\rangle} = 0 \quad \text{for} \quad n \leq [\frac{eb\mathrm{L}}{2\pi}] - 1$$

and denote the normal ordering with respect to $\overline{|\mathrm{vac}; A\rangle}$ by $\vdots \ \vdots$. The exotic matter current operator is

$$\hat{\tilde{j}}^s_{\mathrm{R}}(x) = \hbar \sum_{n \in \mathcal{Z}} \frac{1}{\mathrm{L}} \exp\{i \frac{2\pi}{\mathrm{L}} nx\} \cdot \tilde{\rho}_s(n),$$

$$\tilde{\rho}_s(n) = \sum_{k \in \mathcal{Z}} \frac{1}{2} [\tilde{a}_k^\dagger, \tilde{a}_{k+n}]_- \cdot |\lambda \varepsilon_{k,\mathrm{R}}|^{-s} |\lambda \varepsilon_{k+n,\mathrm{R}}|^{-s}.$$

The new operators $\tilde{\rho}(n)$ and the old ones $\rho(n)$ are connected in the following way:

$$\tilde{\rho}(n) = \rho(n) + \delta_{n,0} [\frac{eb\mathrm{L}}{2\pi}].$$

The total exotic matter charge is

$$\hat{\tilde{\mathrm{Q}}}_{\mathrm{R}} = \hat{\mathrm{Q}}_{\mathrm{R}} + \hbar [\frac{eb\mathrm{L}}{2\pi}].$$

On the physical states 28

$$\hat{\tilde{\mathrm{Q}}}_{\mathrm{R,N}} |\mathrm{phys}; A\rangle \equiv \hbar \vdots \tilde{\rho}(0) \vdots |\mathrm{phys}; A\rangle = \hbar [\frac{eb\mathrm{L}}{2\pi}] |\mathrm{phys}; A\rangle. \tag{53}$$

The old creation and annihilation operators act on the new Fock vacuum by the rule:

$$a_n \overline{|\mathrm{vac}; A\rangle} = 0 \quad \text{for} \quad n > 0,$$
$$a_n^\dagger \overline{|\mathrm{vac}; A\rangle} = 0 \quad \text{for} \quad n \leq 0.$$

If we compare the old and the new Fock vacuum states, then we see a shift of the level that separates the filled levels and the empty ones. The new Fock vacuum is defined such that the levels with energy lower than (or equal to) the energy of the level $n = 0$ are filled and the others are empty, i.e. the background charge is incorporated in the new Fock vacuum.

Using 53, we rewrite $\hat{\mathrm{H}}_{\mathrm{phys}}$ in the compact form with matter and gauge-field degrees of freedom decoupled:

$$\hat{\mathrm{H}}_{\mathrm{phys}} = \hat{\mathrm{H}}_u + \hat{\mathrm{H}}_{\mathrm{matter}},$$

where

$$\hat{\mathrm{H}}_u \equiv \mathrm{L}\{\frac{1}{2\mathrm{L}^2} (\hat{\pi}_u + \hbar \frac{e\mathrm{L}}{2\pi} \theta)^2 + \frac{e^2}{4\pi} \hbar u^2\}$$

is a Hamiltonian governing the dynamics of the fractional part of $\frac{eb\mathrm{L}}{2\pi}$:

$$u \equiv \frac{2\pi}{e\mathrm{L}}(\{\frac{eb\mathrm{L}}{2\pi}\} - \frac{1}{2}), \qquad \hat{\pi}_u \equiv -i\hbar\frac{d}{du},$$

while the matter Hamiltonian is

$$\begin{aligned} \hat{\mathrm{H}}_{\mathrm{matter}} &= \hat{\mathrm{H}}_{(1)} + \hat{\mathrm{H}}_{(2)}, \\ \hat{\mathrm{H}}_{(1)} &\equiv \frac{\hbar}{2}\sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}} \frac{1}{p}\varepsilon_{\mathrm{R}}(p)\rho_{tot}(-p)\rho_{tot}(p), \qquad (54) \\ \hat{\mathrm{H}}_{(2)} &\equiv \hbar^2\frac{e^2\mathrm{L}}{32\pi^2}\sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}}\sum_{\substack{q\in\mathcal{Z}\\ q\neq 0}} \frac{(-1)^{p+q}}{pq}\rho(-p)\rho(q). \end{aligned}$$

The second term $\hat{\mathrm{H}}_{(2)}$ appears after solving the constraint 47. The operators

$$\rho_{tot}(\pm p) \equiv \rho(\pm p) + (-1)^p\frac{1}{\hbar}\hat{\hat{\mathrm{Q}}}_{\mathrm{R,N}}$$

are invariant under both topologically trivial and nontrivial gauge transformations.

Thus, the physical quantum CSM can be formulated in two equivalent ways. In the first way, the matter fields are fermionic and coupled nontrivially to the global gauge-field degree of freedom. In the second way, the matter and gauge-field degrees of freedom are decoupled in the Hamiltonian, but the matter fields acquire exotic statistics. It is the background-matter interaction that leads to exotic statistics of the matter fields. The concepts of background-matter interaction and exotic statistics are therefore closely related.

5 Poincare Algebra

1. The classical momentum and boost generators are given by

$$\mathrm{P_R} = \int dx(-i\hbar\psi_{\mathrm{R}}^{\star}\partial_1\psi_{\mathrm{R}} - \mathrm{E}\partial_1 A),$$

$$\mathrm{K_R} = \int dx \cdot x\mathcal{H}_{\mathrm{R}}.$$

The momentum generator translates A_1 and ψ_{R} in space:

$$\begin{aligned} \{\mathrm{P_R}, A_1(y)\} &= \partial_1 A_1(y), \\ \{\mathrm{P_R}, \psi_{\mathrm{R}}(y)\} &= \partial_1\psi_{\mathrm{R}}(y), \end{aligned}$$

while the boost generator acts as follows

$$\begin{aligned} \{\mathrm{K_R}, A_1(y)\} &= -y\dot{A}_1(y), \\ \{\mathrm{K_R}, \psi_{\mathrm{R}}(y)\} &= -y\dot{\psi}_{\mathrm{R}}(y). \end{aligned}$$

After a straightforward calculation we obtain the algebra

$$\{\mathrm{H_R}, \mathrm{P_R}\} = 0,$$

$$\{\mathrm{P_R}, \mathrm{K_R}\} = -\mathrm{H_R}, \quad \{\mathrm{H_R}, \mathrm{K_R}\} = -\mathrm{P_R},$$

i.e. at the classical level, these generators obey the Poincare algebra.

At the quantum level, the momentum generator becomes

$$\hat{\mathrm{P}}_{\mathrm{R}}^{s} = \frac{1}{2}\hbar\int dx(\psi_{\mathrm{R}}^{\dagger s}(-i\partial_x)\psi_{\mathrm{R}}^{s} - \psi_{\mathrm{R}}^{s}(i\partial_x)\psi_{\mathrm{R}}^{\dagger s}) - \int dx\hat{\mathrm{E}}\partial_1 A_1. \tag{55}$$

Using the Fourier expansions 6, 16 and the quantum Gauss law constraint 27, we rewrite the quantum momentum as

$$\hat{\mathrm{P}}_{\mathrm{R}}^{s} = \hat{\mathrm{H}}_0^{s} - \frac{e^2\mathrm{L}}{2\pi}\hbar\sum_{p>0}\alpha_p\alpha_{-p} - \frac{1}{2}\xi_{\mathrm{R}}\hbar - \frac{1}{2}eb\mathrm{L}\cdot\eta_{\mathrm{R}}\hbar. \tag{56}$$

As the Hamiltonian, the momentum generator is not unique. We act in the same way as before in Section 4 . To get the physical momentum generator, we first define

$$\hat{\hat{\mathrm{P}}}_{\mathrm{R}} \equiv \hat{\mathrm{P}}_{\mathrm{R}} + \frac{1}{2}\sum_{p>0}\{[\hat{v}_{\mathrm{P},+}, \hat{\mathrm{G}}_{+}]_{+} + [\hat{v}_{\mathrm{P},-}, \hat{\mathrm{G}}_{-}]_{+}\}$$

and impose the condition

$$[\hat{\hat{\mathrm{P}}}_{\mathrm{R}}, \hat{\hat{\mathrm{G}}}_{\pm}(p)]_{-} = 0. \tag{57}$$

The condition 57 fix $\hat{v}_{\mathrm{P},\pm}$ and makes the momentum operator invariant under the topologically trivial gauge transformations. Next, we modify $\hat{\tilde{\mathrm{P}}}_{\mathrm{R}}$ by

$$\hat{\tilde{\mathrm{P}}}_{\mathrm{R}} \to \hat{\tilde{\mathrm{P}}}_{\mathrm{R}} + \beta_{\mathrm{P},0} + \sum_{p>0}(\beta_{\mathrm{P},+}(p)\rho(p) + \beta_{\mathrm{P},-}(p)\rho(-p)) \tag{58}$$

in order to make it invariant under the nontrivial gauge transformations as well:

$$[\hat{\tilde{\mathrm{P}}}_{\mathrm{R}}, \hat{\mathrm{T}}_b]_- = 0. \tag{59}$$

Finding $\hat{v}_{\mathrm{P},\pm}$ from Eq. 57 and $(\beta_{\mathrm{P},0}, \beta_{\mathrm{P},\pm})$ from Eq. 59 and substituting them into 58, we get the physical quantum momentum in the form

$$\hat{\tilde{\mathrm{P}}}_{\mathrm{R}}|\mathrm{phys}; A\rangle = \hat{\mathrm{P}}_{\mathrm{phys}}|\mathrm{phys}; A\rangle,$$

$$\hat{\mathrm{P}}_{\mathrm{phys}} = \hat{\mathrm{P}}_{\mathrm{matter}} = \frac{\pi}{\mathrm{L}}\hbar \sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}} \rho_{\mathrm{tot}}(-p)\rho_{\mathrm{tot}}(p). \tag{60}$$

The action of this operator on the matter fields $\rho(\pm p)$ is

$$[\hat{\mathrm{P}}_{\mathrm{phys}}, \rho_{\mathrm{tot}}(\pm p)]_- = \mp\frac{2\pi}{\mathrm{L}}\hbar p \cdot \rho_{\mathrm{tot}}(\pm p).$$

The quantum boost generator is

$$\hat{\mathrm{K}}_{\mathrm{R}} = -i\hbar\frac{\mathrm{L}}{2\pi}\sum_{p>0}\frac{(-1)^p}{p}(\mathcal{H}_{\mathrm{R}}(p) - \mathcal{H}_{\mathrm{R}}(-p)),$$

where

$$\mathcal{H}_{\mathrm{R}}(p) = \mathcal{H}_{\mathrm{F}}(p) + \mathcal{H}_{\mathrm{EM}}(p),$$

$\mathcal{H}_{\mathrm{F}}(p)$ and $\mathcal{H}_{\mathrm{EM}}(p)$ being given by Eqs. 12 and 17 correspondingly.

The physical quantum boost generator can be constructed in the same way as the physical Hamiltonian and momentum and has the form

$$\hat{\mathrm{K}}_{\mathrm{phys}} = \hat{\mathrm{K}}_{\mathrm{matter}} = \hat{\mathrm{K}}_{(1)} + \hat{\mathrm{K}}_{(2)},$$

$$\hat{\mathrm{K}}_{(1)} \equiv -\frac{i\hbar}{2}\sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}}\frac{(-1)^p}{p}\sum_{\substack{q\in\mathcal{Z}\\ q\neq(0;-p)}} \mathrm{k}_{\mathrm{R}}(p,q)\cdot\rho_{\mathrm{tot}}(-q)\rho_{\mathrm{tot}}(p+q), \tag{61}$$

$$\hat{\mathrm{K}}_{(2)} = \frac{\hbar}{8\pi}(\frac{e\mathrm{L}}{\pi})^2\sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}}\frac{(-1)^p}{p^2}\rho_{\mathrm{tot}}(p)\cdot\hat{\pi}_{[\frac{eb\mathrm{L}}{2\pi}]},$$

where

$$\mathrm{k}_{\mathrm{R}}(p,q) \equiv 1 + \frac{\hbar}{8\pi}(\frac{e\mathrm{L}}{\pi})^2\frac{1}{q(q+p)}.$$

On the states 47 $\hat{\mathrm{K}}_{(2)}$ becomes

$$\hat{\mathrm{K}}_{(2)} = i\frac{\hbar^2}{16\pi}(\frac{e\mathrm{L}}{\pi})^2 \sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}} \frac{(-1)^p}{p^2} \sum_{\substack{q\in\mathcal{Z}\\ q\neq 0}} \frac{(-1)^q}{q} \rho(-q)\rho_{\mathrm{tot}}(p).$$

2. Let us now construct the algebra of the physical Hamiltonian, momentum and boost generators. Since the relativistic invariance is broken, this algebra is not definetely a Poincare one. We neglect, for the moment, the global gauge-field degree of freedom contribution and start with the following generators :

$$\begin{aligned}
\hat{\mathrm{H}}_{\mathrm{phys}} &= \sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}} \mathrm{H}(p),\\
\hat{\mathrm{P}}_{\mathrm{phys}} &= \sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}} \mathrm{P}(p),\\
\hat{\mathrm{K}}_{\mathrm{phys}} &= \sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}} \sum_{\substack{q\in\mathcal{Z}\\ q\neq(0,-p)}} \mathrm{K}(p,q),
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{H}(p) &\equiv \frac{\hbar}{2}\frac{1}{p}\varepsilon_{\mathrm{R}}(p)\rho(-p)\rho(p),\\
\mathrm{P}(p) &\equiv \hbar\frac{\pi}{\mathrm{L}}\rho(-p)\rho(p),\\
\mathrm{K}(p,q) &\equiv -\frac{i\hbar}{2}\frac{(-1)^p}{p}\mathrm{k}_{\mathrm{R}}(p,q)\rho(-q)\rho(p+q).
\end{aligned}$$

We can check by a straightforward calculation that

$$[\hat{\mathrm{H}}_{\mathrm{phys}}, \hat{\mathrm{P}}_{\mathrm{phys}}]_- = 0,$$

i.e. the translational invariance is preserved , while two other commutation relations

$$[\hat{\mathrm{P}}_{\mathrm{phys}}, \hat{\mathrm{K}}_{\mathrm{phys}}]_- \neq -i\hbar\hat{\mathrm{H}}_{\mathrm{phys}},$$

$$[\hat{\mathrm{H}}_{\mathrm{phys}}, \hat{\mathrm{K}}_{\mathrm{phys}}]_- \neq -i\hbar\hat{\mathrm{P}}_{\mathrm{phys}}$$

differ from those of Poincare algebra. In terms of $\mathrm{H}(p), \mathrm{P}(p), \mathrm{K}(p,q)$ (p, q are nonzero) these commutation relations are written in a compact form as follows

$$[\mathrm{H}(p), \mathrm{K}(q,m)]_- = \frac{\hbar}{2}\varepsilon_{\mathrm{R}}(p)\{\mathrm{K}(q,p)\cdot(\delta_{m,-p-q} + \delta_{m,p}) - (p \to -p)\}, \quad q \neq \pm p,$$

$$[\mathrm{H}(p), \mathrm{K}(\pm p, m)]_- = \pm\frac{\hbar}{2}\varepsilon_{\mathrm{R}}(p)\cdot\mathrm{K}(\pm p, \pm p)(\delta_{m,\mp 2p} + \delta_{m,\pm p})$$

and

$$[\mathrm{P}(p), \mathrm{K}(q,m)]_- = \hbar\frac{\pi}{\mathrm{L}}p\{\mathrm{K}(q,p)\cdot(\delta_{m,-p-q}+\delta_{m,p}) - (p\to -p)\}, \quad q\neq \pm p,$$

$$[\mathrm{P}(p), \mathrm{K}(\pm p,m)]_- = \pm\hbar\frac{\pi}{\mathrm{L}}p\mathrm{K}(\pm p,\pm p)\cdot(\delta_{m,\mp 2p}+\delta_{m,\pm p}).$$

If we introduce

$$b^{\pm}(p) \equiv \frac{1}{\sqrt{p}}\rho(\mp p), \quad p>0,$$

then

$$\begin{aligned}
\hat{\mathrm{H}}_{\mathrm{phys}} &= \hbar\sum_{p>0}\varepsilon_{\mathrm{R}}(p)b^+(p)b^-(p),\\
\hat{\mathrm{P}}_{\mathrm{phys}} &= \hbar\sum_{p>0}\frac{2\pi}{\mathrm{L}}pb^+(p)b^-(p).
\end{aligned}$$

Therefore, $b^+(p)$ and $b^-(p)$ can be interpreted respectively as the creation and annihilation operators for a particle of momentum $\hbar\frac{2\pi}{\mathrm{L}}p$ and energy $\hbar\varepsilon_{\mathrm{R}}(p)$.

If the global gauge-field degree of freedom contribution is taken into account, then the translational invariance is also lost. Indeed, the total matter Hamiltonian 54 is not invariant under translations, since the second term

$$\hat{\mathrm{H}}_{(2)} = \sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}}\sum_{\substack{q\in\mathcal{Z}\\ q\neq 0}}\mathrm{H}(p,q),$$

$$\mathrm{H}(p,q) \equiv \hbar^2\frac{e^2\mathrm{L}}{32\pi^2}\frac{(-1)^{p+q}}{pq}\rho(-p)\rho(q)$$

does not commute with the physical momentum:

$$\begin{aligned}
[\mathrm{H}(p,q), \mathrm{P}(m)]_- = \frac{\hbar^3 e^2}{32\pi}\frac{(-1)^{p+q}}{pq}&\{p\rho(-q)\rho_{\mathrm{tot}}(p)\cdot(\delta_{p,-m}+\delta_{p,m})\\
&-q\rho_{\mathrm{tot}}(-q)\rho(p)\cdot(\delta_{q,-m}+\delta_{q,m})\}.
\end{aligned}$$

All three commutators of the Poincare algebra are therefore broken. The spectrum of the model is nonrelativistic, and there is no mass in this spectrum.

3. In the limit $\mathrm{L}\to\infty$, when the model is defined on the line R^1 , b vanishes and the gauge field does not possess any physical degree of freedom.

The physical Hamiltonian and momentum commute. Two other commutation relations of the Poincare algebra are broken. As before with L finite , the reason for the breaking of the relativistic invariance is anomaly or , more exactly, the fact that the local gauge symmetry is realized projectively.

For $\mathrm{L}\to\infty$, we can however construct the states which are simultaneous eigenstates of the Hamiltonian and momentum. The corresponding eigenvalues are connected in a relativistic way and allow us to interpret these states as massive [11].

6 Charge Screening

Let us introduce a pair of external charges, namely, a positive charge with strength q at x_0 and a negative one with the same strength at y_0. The external current density is

$$j_{\mathrm{ex},0}(x) = q(\delta(x-x_0) - \delta(x-y_0)) = \frac{1}{\mathrm{L}} \sum_{p \in \mathcal{Z}} j_p^{\mathrm{ex}} \exp\{-i\frac{2\pi p}{\mathrm{L}} x\},$$

where

$$j_p^{\mathrm{ex}} \equiv q(e^{i2\pi p x_0/\mathrm{L}} - e^{i2\pi p y_0/\mathrm{L}}).$$

The total external charge is zero, so the external current density has vanishing zero mode, $j_0^{\mathrm{ex}} = 0$. The Lagrangian density of the CSM changes as follows

$$\mathcal{L} \longrightarrow \mathcal{L} + eA_0 \cdot j_{\mathrm{ex},0}.$$

The classical CSM with the external charges added can be quantized in the same way as that without external charges. The quantum Gauss' law operator becomes

$$\hat{\mathrm{G}}_{\mathrm{ex}} \equiv \hat{\mathrm{G}} + e j_{\mathrm{ex},0} = \partial_1 \hat{\mathrm{E}} + e(\hat{j}_{\mathrm{R}} + j_{\mathrm{ex},0}).$$

Its Fourier expansion is

$$\hat{\mathrm{G}}_{\mathrm{ex}} = \hat{\mathrm{G}}_0 + \frac{2\pi}{\mathrm{L}^2} \sum_{p>0} (\hat{\mathrm{G}}_{+}^{\mathrm{ex}}(p) e^{i\frac{2\pi}{\mathrm{L}}px} - \hat{\mathrm{G}}_{-}^{\mathrm{ex}}(p) e^{-i\frac{2\pi}{\mathrm{L}}px}),$$

where

$$\begin{aligned} \hat{\mathrm{G}}_{+}^{\mathrm{ex}}(p) &\equiv \hat{\mathrm{G}}_{+}(p) + \frac{e\mathrm{L}}{2\pi}(j_p^{\mathrm{ex}})^{\star}, \\ \hat{\mathrm{G}}_{-}^{\mathrm{ex}}(p) &\equiv \hat{\mathrm{G}}_{-}(p) - \frac{e\mathrm{L}}{2\pi} j_p^{\mathrm{ex}}. \end{aligned}$$

The physical states $|\mathrm{phys}; A; \mathrm{ex}\rangle$ are defined as

$$\hat{\tilde{\mathrm{G}}}_{\pm}^{\mathrm{ex}}(p)|\mathrm{phys}; A; \mathrm{ex}\rangle \equiv (\hat{\mathrm{G}}_{\pm}^{\mathrm{ex}}(\mathrm{p}) \pm \hbar \frac{\mathrm{e}^2\mathrm{L}^2}{8\pi^2} \alpha_{\pm\mathrm{p}})|\mathrm{phys}; \mathrm{A}; \mathrm{ex}\rangle = 0.$$

The external charges change also the Fock vacuum. We have the following definition for the Fock vacuum in the presence of the external charges:

$$\begin{aligned} (\rho(p) + \frac{1}{\hbar}(j_p^{\mathrm{ex}})^{\star})|\mathrm{vac}; A; \mathrm{ex}\rangle &= 0, \\ \langle \mathrm{ex}; \mathrm{vac}; A|(\rho(-p) + \frac{1}{\hbar} j_p^{\mathrm{ex}}) &= 0, \quad \text{for} \quad p > 0. \end{aligned} \tag{62}$$

The physical quantum matter Hamiltonian invariant under the both topologically trivial and nontrivial gauge transformations becomes

$$\hat{\mathrm{H}}_{(1)} = \frac{\pi}{\mathrm{L}} \hbar \sum_{\substack{p \in \mathcal{Z} \\ p \neq 0}} \rho_{\mathrm{tot}}(-p) \rho_{\mathrm{tot}}(p) + \frac{e^2\mathrm{L}}{8\pi^2} \hbar^2 \sum_{\substack{p \in \mathcal{Z} \\ p \neq 0}} \frac{1}{p^2} (\rho(-p) + \frac{1}{\hbar} j_p^{\mathrm{ex}}) \cdot (\rho(p) + \frac{1}{\hbar}(j_p^{\mathrm{ex}})^{\star}),$$

$\hat{\mathrm{H}}^{\mathrm{phys}}_{\mathrm{matter},(2)}$ being given again by Eq. 54.

We consider two different cases. 1. Let us neglect the global gauge-field degree of freedom contribution to the matter Hamiltonian. After some calculations we rewrite it as

$$\hat{\mathrm{H}}_{\mathrm{matter}} = \frac{\hbar}{2}\sum_{\substack{p\in\mathcal{Z}\\ p\neq 0}}\frac{1}{p}\varepsilon_{\mathrm{R}}(p)\rho_{\mathrm{ex}}(-p)\rho_{\mathrm{ex}}(p) + \frac{e^2}{2\pi}\sum_{p>0}\frac{1}{p\varepsilon_{\mathrm{R}}(p)}(j_p^{\mathrm{ex}})^{\star}(j_p^{\mathrm{ex}}), \tag{63}$$

where

$$\begin{aligned}\rho_{\mathrm{ex}}(p) &\equiv \rho(p) + \frac{e^2\mathrm{L}}{4\pi^2}\frac{1}{p\varepsilon_{\mathrm{R}}(p)}(j_p^{\mathrm{ex}})^{\star},\\ \rho_{\mathrm{ex}}(-p) &\equiv \rho(-p) + \frac{e^2\mathrm{L}}{4\pi^2}\frac{1}{p\varepsilon_{\mathrm{R}}(p)}j_p^{\mathrm{ex}}.\end{aligned} \tag{64}$$

The ground state of this Hamiltonian differs from the vacuum one 62 and is defined as

$$\begin{aligned}\rho_{\mathrm{ex}}(p)|\mathrm{ground};\mathrm{ex}\rangle &= 0,\\ \langle\mathrm{ex};\mathrm{ground}|\rho_{\mathrm{ex}}(-p) &= 0, \quad p>0.\end{aligned}$$

The first term in 63 is normal ordered with respect to this state and the second one is its energy:

$$\mathrm{E}_0 = \langle\mathrm{ground};\mathrm{ex}|\hat{\mathrm{H}}_{\mathrm{matter}}|\mathrm{ground};\mathrm{ex}\rangle = \frac{\mathrm{e}^2}{2\pi}\sum_{\mathrm{p}>0}\frac{1}{\mathrm{p}\varepsilon_{\mathrm{R}}(\mathrm{p})}(\mathrm{j}_{\mathrm{p}}^{\mathrm{ex}})^{\star}(\mathrm{j}_{\mathrm{p}}^{\mathrm{ex}}).$$

The energy E_0 depends only on the distance between the external charges:

$$\begin{aligned}\mathrm{E}_0 &= 4\frac{(eq)^2}{\mathrm{L}}\sum_{p>0}\frac{1}{(\frac{2\pi p}{\mathrm{L}})^2+\frac{e^2}{2\pi}\hbar}\{1-\cos(\frac{2\pi p}{\mathrm{L}}(x_0-y_0))\}\\ &= \frac{(eq)^2}{m_0}\frac{\cosh\frac{\mathrm{L}m_0}{2}-\cosh(\frac{\mathrm{L}m_0}{2}-m_0|x_0-y_0|)}{\sinh\frac{\mathrm{L}m_0}{2}},\end{aligned}$$

where $m_0^2 = \frac{e^2}{2\pi}\hbar$. In the limit $\mathrm{L}\gg 1$, we get

$$\mathrm{E}_0 = \frac{(eq)^2}{m_0}(1-e^{-m_0|x_0-y_0|}),$$

i.e. the Yukawa potential.

The current density induced by the two external charges is

$$\langle\mathrm{ground};\mathrm{ex}|\hat{\mathrm{j}}_{\mathrm{R}}(\mathrm{x})|\mathrm{ground};\mathrm{ex}\rangle \equiv \mathrm{f}(\mathrm{x},\mathrm{x}_0)-\mathrm{f}(\mathrm{x},\mathrm{y}_0),$$

where

$$f(x,x_0) \equiv -\frac{e^2q}{2\pi^2}\hbar\sum_{p>0}\frac{1}{p\varepsilon_{\mathrm{R}}(p)}\cos(\frac{2\pi p}{\mathrm{L}}(x-x_0)) = -\frac{qm_0}{2}\frac{\cosh(\frac{\mathrm{L}m_0}{2}-m_0|x-x_0|)}{\sinh\frac{\mathrm{L}m_0}{2}}.$$

The induced current density is a sum of the current densities induced by the each charge. In the limit $\mathrm{L} \gg 1$,

$$f(x, x_0) \simeq -\frac{qm_0}{2} e^{-m_0|x-x_0|}$$

and damps exponentially as x goes far from x_0, so the external charges are screened globally. If we are far away from the external charges , we can not find them.

2. Let us now take into account the gauge-field contribution and consider the total matter Hamiltonian . We can diagonalize it in the following form

$$\hat{\mathrm{H}}_{\mathrm{matter}} = \hbar \sum_{\substack{p\in\mathcal{Z} \\ p\neq 0}} \varepsilon_{\mathrm{R}}(p)\tilde{\rho}_{\mathrm{ex}}(-p)\tilde{\rho}_{\mathrm{ex}}(p) + \hbar^2 \frac{e^2\mathrm{L}}{32\pi^2} \sum_{\substack{p\in\mathcal{Z} \\ p\neq 0}} \sum_{\substack{q\in\mathcal{Z} \\ q\neq 0}} \frac{(-1)^{p+q}}{pq} \tilde{\rho}_{\mathrm{ex}}(-p)\tilde{\rho}_{\mathrm{ex}}(q) + \tilde{\mathrm{E}}_0(x_0, y_0; b),$$

where

$$\tilde{\rho}_{\mathrm{ex}}(\pm p) \equiv \rho_{\mathrm{ex}}(\pm p) + (-1)^p[\frac{eb\mathrm{L}}{2\pi}] \pm iq\hbar \frac{e^4\mathrm{L}}{16\pi^3} \frac{(-1)^p}{\varepsilon_{\mathrm{R}}(p)} \cdot \frac{d_2}{1+\frac{e^2}{4\pi}\hbar d_1},$$

with $\rho_{\mathrm{ex}}(\pm p)$ given by Eq. 64, and

$$d_1 \equiv \frac{\mathrm{L}}{4m_0} \cdot \frac{\cosh\frac{\mathrm{L}m_0}{2} - \cosh\frac{\mathrm{L}m_0}{4}}{\sinh\frac{\mathrm{L}m_0}{2}},$$

$$d_2 \equiv \frac{\pi}{2m_0} \cdot \frac{m_0(\sinh m_0x_0 - \sinh m_0y_0) + (x_0 - y_0)\cosh\frac{\mathrm{L}m_0}{4}}{\sinh\frac{\mathrm{L}m_0}{2}}.$$

The ground state of the total matter Hamiltonian satisfies

$$\begin{aligned} \tilde{\rho}_{\mathrm{ex}}(p)\overline{|\mathrm{ground};\mathrm{ex}\rangle} &= 0, \\ \overline{\langle\mathrm{ex};\mathrm{ground}|}\tilde{\rho}_{\mathrm{ex}}(-p) &= 0, \quad \text{for} \quad p > 0. \end{aligned}$$

The energy of the ground state is

$$\tilde{\mathrm{E}}_0(x_0, y_0; b) = \overline{\langle\mathrm{ground};\mathrm{ex}|}\hat{\mathrm{H}}_{\mathrm{matter}}\overline{|\mathrm{ground};\mathrm{ex}\rangle}$$

$$= \mathrm{E}_0(x_0 - y_0) + \hbar q \frac{e^2}{2\mathrm{L}}[\frac{eb\mathrm{L}}{2\pi}](x_0^2 - y_0^2) + \hbar^2 q^2 \frac{e^6\mathrm{L}}{32\pi^4} \cdot \frac{d_2^2}{1+\frac{e^2}{4\pi}\hbar d_1}$$

(up to constants independent of x_0 and y_0). In contrast with $\mathrm{E}_0(x_0 - y_0)$, this energy depends not only on the distance between the external charges, but also separately on x_0 and y_0. For $\mathrm{L} \gg 1$, we have

$$\tilde{\mathrm{E}}_0(x_0, y_0; b) \simeq \frac{(eq)^2}{m_0}(1 - e^{-m_0|x_0-y_0|}) + \hbar q \frac{e^2}{2\mathrm{L}}[\frac{eb\mathrm{L}}{2\pi}](x_0^2 - y_0^2) + \hbar q^2 \frac{e^4}{8\pi m_0}(x_0 - y_0)^2.$$

The induced current density is

$$\overline{\langle\mathrm{ground};\mathrm{ex}|}\hat{j}_R(x)\overline{|\mathrm{ground};\mathrm{ex}\rangle} = \tilde{f}(x; x_0) - \tilde{f}(x; y_0) - \frac{1}{\mathrm{L}}\hbar[\frac{eb\mathrm{L}}{2\pi}], \tag{65}$$

where

$$\tilde{f}(x;x_0) = f(x;x_0) - \frac{e^4 q \mathrm{L}}{64\pi}\frac{1}{m_0}\hbar^2 \frac{\sinh m_0 x}{(\sinh \frac{\mathrm{L}m_0}{2})^2}(m_0 \sinh m_0 x_0 + x_0 \cosh \frac{\mathrm{L}m_0}{4}).$$

The last term in 65 is the same for all values of x and induced by the global gauge-field degree of freedom. In the limit $\mathrm{L} \gg 1$, we have

$$\tilde{f}(x,x_0) \simeq -\frac{q}{2}(m_0 e^{-m_0|x-x_0|} + \hbar \frac{e^2}{2\pi} x_0 e^{-\frac{3}{4}\mathrm{L}m_0} \sinh m_0 x).$$

The second term here is very small for large, but finite L. At the same time, it increases exponentially when x goes to infinity. The external charges are not therefore screened even globally.

7 Discussion

We have shown that the anomaly influences essentially the physical quantum picture of the CSM. For the model defined on S^1, when the gauge field has a global physical degree of freedom, the left– right asymmetric matter content results in the background linearly rising electric field or ,equivalently, in the exotic statistics of the physical matter field. This is a new physical effect caused just by the anomaly and absent in the standard Schwinger model.

The anomaly leads also to the breaking of the relativistic invariance. We have constructed the Poincare generators and shown that their algebra is not a Poincare one. The spectrum of the physical Hamiltonian is not relativistic and does not contain a massive boson.

Next, the external charges are not screened. Owing to the global gauge-field degree of freedom contribution to the physical Hamiltonian, the current density induced by the external charges doers not vanish globally. Thus, such phenomena as the dynamical mass generation and the total screening of charges characteristic for the Schwinger model do not take place for the CSM on S^1.

For the CSM defined on R^1, the physical quantum picture differs from that on S^1. The gauge field has not any physical degree of freedom, and the background electric field disappears. The current density induced by the external charges damps exponentially far away from them. The external charges are then globally screened.

The anomaly manifests itself only in the breaking of the relativistic invariance. However, the theory is invariant under space translations. As shown in [11], [12], this allows us to construct the massive states which are simultaneous eigenstates of the physical Hamiltonian and momentum. The screening of the external charges and the dynamical mass generation (although in a different way) are therefore valid for the physical quantum CSM on R^1.

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Appendix

i) In this appendix we prove the commutation relations 18 – 20. We start with the commutation relation 18. It can be established in different ways [19, 28]. Here we derive it using the ζ–function regularization scheme. For the regulated operators $\rho_s(m)$ we get

$$[\rho_s(m), \rho_s(n)]_- = \sum_{k\in\mathcal{Z}} \frac{1}{2}[a_k^\dagger, a_{k+m+n}]_- \cdot |\lambda\varepsilon_{k,\mathrm{R}}|^{-s/2} |\lambda\varepsilon_{k+m+n,\mathrm{R}}|^{-s/2}$$

$$\cdot(|\lambda\varepsilon_{k+m,\mathrm{R}}|^{-s} - |\lambda\varepsilon_{k+n,\mathrm{R}}|^{-s}).$$

Since the commutator $[\rho(m), \rho(n)]_-$ is a C–number, we calculate it by taking the corresponding vacuum expectation value:

$$\langle \mathrm{vac}; A|[\rho_s(m), \rho_s(n)]_-|\mathrm{vac}; A\rangle = \delta_{m,-n}\mathrm{I}_1^s(m),$$

where

$$\mathrm{I}_1^s(m) \equiv -\frac{1}{2}\sum_{k\in\mathcal{Z}} \mathrm{sign}(\varepsilon_k)|\lambda\varepsilon_{k,\mathrm{R}}|^{-s}(|\lambda\varepsilon_{k+m,\mathrm{R}}|^{-s} - |\lambda\varepsilon_{k-m,\mathrm{R}}|^{-s}).$$

We see that the commutator is nonvanishing only for $m = -n$. The sum $\mathrm{I}_1^s(m)$ can be easily evaluated. In particular, for $m > 0$, we have

$$\sum_{k\in\mathcal{Z}} \mathrm{sign}(\varepsilon_k)|\lambda\varepsilon_{k,\mathrm{R}}|^{-s}|\lambda\varepsilon_{k\pm m,\mathrm{R}}|^{-s}$$

$$= \sum_{k>0} \frac{1}{(k - \{\frac{eb\mathrm{L}}{2\pi}\})^s(k - \{\frac{eb\mathrm{L}}{2\pi}\} + m)^s} - \sum_{k\geq 0} \frac{1}{(k + \{\frac{eb\mathrm{L}}{2\pi}\})^s(k + \{\frac{eb\mathrm{L}}{2\pi}\})^s} \mp m,$$

so $\mathrm{I}_1^s(m) = m$ for all values of s and

$$[\rho(m), \rho(n)]_- = \lim_{s\to 0}[\rho_s(m), \rho_s(n)]_- = m\delta_{m,-n}.$$

ii) Let us now calculate the derivatives $\frac{d}{db}\rho(m)$ and $\frac{d}{d\alpha_{\pm p}}\rho(m)$. With Eq. 9, we have

$$\frac{d}{db}\rho_s(m) = \frac{1}{2}\sum_{n\in\mathcal{Z}}\sum_{k\in\mathcal{Z}}(\langle n; \mathrm{R}|\frac{d}{db}|k; \mathrm{R}\rangle \cdot [a_n^\dagger, a_{k+m}]_- - \langle k+m; \mathrm{R}|\frac{d}{db}|n; \mathrm{R}\rangle \cdot [a_k^\dagger, a_n]_-)$$

$$\cdot|\lambda\varepsilon_{k,\mathrm{R}}|^{-s/2}|\lambda\varepsilon_{n,\mathrm{R}}|^{-s/2}|\lambda\varepsilon_{k+m,\mathrm{R}}|^{-s/2}$$

and

$$\frac{d}{d\alpha_{\pm p}}\rho_s(m) = \frac{1}{2}\sum_{n\in\mathcal{Z}}\sum_{k\in\mathcal{Z}}(\langle n; \mathrm{R}|\frac{\mathrm{d}}{\mathrm{d}\alpha_{\pm \mathrm{p}}}|\mathrm{k}; \mathrm{R}\rangle \cdot [\mathrm{a}_\mathrm{n}^\dagger, \mathrm{a}_{\mathrm{k+m}}]^- - \langle \mathrm{k+m}; \mathrm{R}|\frac{\mathrm{d}}{\mathrm{d}\alpha_{\pm \mathrm{p}}}|\mathrm{n}; \mathrm{R}\rangle \cdot [\mathrm{a}_\mathrm{k}^\dagger, \mathrm{a}_\mathrm{n}]_-)$$

$$\cdot|\lambda\varepsilon_{k,\mathrm{R}}|^{-s/2}|\lambda\varepsilon_{n,\mathrm{R}}|^{-s/2}|\lambda\varepsilon_{k+m,\mathrm{R}}|^{-s/2}.$$

Substituting

$$\langle n;\mathrm{R}|\frac{\mathrm{d}}{\mathrm{db}}|\mathrm{k};\mathrm{R}\rangle = \frac{\mathrm{ie}}{2}\mathrm{L}\delta_{\mathrm{n,k}},$$

$$\langle n;\mathrm{R}|\frac{\mathrm{d}}{\mathrm{d}\alpha_{\pm\mathrm{p}}}|\mathrm{k};\mathrm{R}\rangle = \pm\frac{\mathrm{eL}}{2\pi\mathrm{p}}\delta_{\mathrm{k-n,\mp p}} \mp \frac{\mathrm{eL}}{2\pi}\frac{(-1)^{\mathrm{p}}}{\mathrm{p}}\delta_{\mathrm{n,k}}$$

into these equations and taking again the vacuum expactation values , we obtain

$$\langle \mathrm{vac};A|\frac{d}{db}\rho_s(m)|\mathrm{vac};A\rangle = 0,$$

$$\langle \mathrm{vac};A|\frac{d}{d\alpha_{\pm p}}\rho_s(m)|\mathrm{vac};A\rangle = -\frac{e\mathrm{L}}{4\pi p}\delta_{m,\pm p}\cdot \mathrm{I}_2^s(m),$$

where

$$\mathrm{I}_2^s(m) \equiv \sum_{k\in\mathcal{Z}} \mathrm{sign}(\varepsilon_{k,\mathrm{R}})|\lambda\varepsilon_{k,\mathrm{R}}|^{-s}(|\lambda\varepsilon_{k-p,\mathrm{R}}|^{-s/2} - |\lambda\varepsilon_{k+p,\mathrm{R}}|^{-s/2}).$$

For large s, $\mathrm{I}_2^s(m) \simeq 2\mathrm{I}_1^s(m) = 2m$, so we finally come to the Eqs. 20.

iii) To prove 19, we calculate the commutator of the corresponding regulated operators:

$$[\hat{\mathrm{H}}_0^s,\rho_s(p)]_- = \frac{2\pi}{\mathrm{L}}\hbar\sum_{k\in\mathcal{Z}}\frac{k}{2}[a_k^\dagger, a_{k+p}]_-\cdot|\lambda\varepsilon_{k,\mathrm{R}}|^{-3s/2}|\lambda\varepsilon_{k+p,\mathrm{R}}|^{-s/2}$$

$$-\frac{2\pi}{\mathrm{L}}\hbar\sum_{k\in\mathcal{Z}}\frac{k}{2}[a_{k-p}^\dagger, a_k]_-\cdot|\lambda\varepsilon_{k,\mathrm{R}}|^{-3s/2}|\lambda\varepsilon_{k-p,\mathrm{R}}|^{-s/2}, \qquad (p>0).$$

If we make the redefinition $k-p\to k$ in the second sum, then

$$[\hat{\mathrm{H}}_0^s,\rho_s(p)]_- = \frac{2\pi}{\mathrm{L}}\hbar\sum_{k\in\mathcal{Z}}\frac{k}{2}[a_k^\dagger, a_{k+p}]_-\cdot|\lambda\varepsilon_{k,\mathrm{R}}|^{-s/2}|\lambda\varepsilon_{k+p,\mathrm{R}}|^{-s/2}(|\lambda\varepsilon_{k,\mathrm{R}}|^{-s} - |\lambda\varepsilon_{k+p,\mathrm{R}}|^{-s})$$

$$-\frac{2\pi}{\mathrm{L}}\hbar p\sum_{k\in\mathcal{Z}}\frac{1}{2}[a_k^\dagger, a_{k+p}]_-\cdot|\lambda\varepsilon_{k,\mathrm{R}}|^{-s/2}|\lambda\varepsilon_{k+p,\mathrm{R}}|^{-3s/2}.$$

For large s, the first term vanishes, so

$$[\hat{\mathrm{H}}_0^s,\rho_s(p)]_- = -\frac{2\pi}{\mathrm{L}}\hbar p\rho_s(p)|\lambda\varepsilon_{p,\mathrm{R}}|^{-s}.$$

In the limit $s\to 0$, we then get 19. Similarly, for $\rho(-p)$ we have

$$[\hat{\mathrm{H}}_0^s,\rho_s(-p)]_- = \frac{2\pi}{\mathrm{L}}\hbar p\rho_s(-p)|\lambda\varepsilon_{-p,\mathrm{R}}|^{-s}.$$

It can be checked that the bosonized form of $\hat{\mathrm{H}}_0^s$ which reproduces the last two equations is

$$\hat{\mathrm{H}}_0^s = \frac{2\pi}{\mathrm{L}}\hbar\sum_{p>0}|\lambda\varepsilon_{p,\mathrm{R}}|^{-s}\rho_s(p)\rho_s(-p).$$

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